GEODESIC STRING COUNTING INVARIANTS OF MANIFOLDS

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ABSTRACT. We study $\mathbb{Q}$-valued metric/topological invariants of manifolds, by counting closed geodesics, and using the Fuller index. As one application, let g be a regular Finsler metric on T^n sufficiently nearby to the standard flat metric. Let o be a closed geodesic in a non-trivial class β, s.t. a given prime p divides multiplicity of o is even. Then there is at least one other such o. This holds for any regular g with finitely many class β orbits, provided a certain conjectural topological invariance of our geodesic string count holds. The formulation extends from metrics to Reeb vector fields on the unit cotangent bundle of X. The conjecture is partially verified, and this plays a role for a number of other applications. Along the way, we also prove that sky catastrophes of smooth dynamical systems are not geodesible by a certain class of forward complete Riemann-Finsler metrics, in particular by complete Riemannian metrics with non-positive sectional curvature. This partially answers a question of Fuller and gives further examples for our theory here.

1. Introduction

Using counts of ***geodesic strings*** (equivalence classes of closed, constant speed geodesics up to reparametrization S^1 action), and the Fuller index [7], we will define certain rational number valued, deformation invariants for complete Riemann-Finsler manifolds. A basic case is a complete Riemannian manifold with non-positive sectional curvature, and in this case we get deformation invariants of the metric, provided the deformation is through non-positive sectional curvature metrics. Studying connections of the Fuller index with Riemann-Finsler geometry was suggested by Fuller himself in the 1960's.

These invariants can be directly interpreted as the untwisted part of certain elliptic Gromov-Witten invariants in an associated lcs manifold, [15]. The twist is coming by way of metric isometries.

We also get a conjectural topological invariant of manifolds, the conjecture will be partially verified. See Conjecture 1, and Theorem 1.20. The latter is an important ingredient for some geometric applications here.

Assuming full topological invariance of our string counting invariant we get the following perhaps mysterious theorem. For a non-constant class β we say that a metric g on X is β-***regular*** if all of its class β closed geodesics are non-degenerate the usual sense, or equivalently the closed orbits of the associated geodesic flow are dynamically non-degenerate. For a geodesic string o, $\operatorname{mult}(o)$ will denote its multiplicity, that is the order of the corresponding isotropy subgroup of S^1, where S^1 is acting by reparametrization.

Theorem 1.1.

Assume the Conjecture 1 and let g be a β-regular Finsler metric on T^n having finitely many β class geodesic strings. Suppose that o is a g-geodesic string s.t.:

1. *o has class β.*
2. *$p | \operatorname{mult}(o)$, where p is prime,*

Key words and phrases. locally conformally symplectic manifolds, Gromov-Witten theory, virtual fundamental class, Fuller index.

is even,

then there is at least one other such o.

Remark 1.2. We can extend the above from metrics to Reeb vector fields on the unit cotangent bundle, representing the standard contact structure. Now counting closed Reeb orbits in classes $\widetilde{\beta}$, lifting a class β as above. But we need a stronger form of the Conjecture 1, see Remark 1.17. The following local result, Theorem 1.3, also holds in the Reeb context, with basically the same proof.

Without Conjecture 1 we have a local form of the above.

Theorem 1.3. *Let g_0 be the standard flat metric on T^n. For all $\epsilon > 0$ sufficiently small, whenever g' is a Finsler metric C^0 ϵ close to g_0 and is β-regular, the following holds. Suppose g' has a closed geodesic string o s.t.*

(1) *o has class β.*

(2) *$p|\operatorname{mult}(o)$, where p is prime.*

Then g' has at least one other geodesic string satisfying these conditions. Moreover, any Finsler metric g_1 taut deformation equivalent to g_0 (see Definition 1.10) has the same property as g_0, above.

Recall that a now classical theorem of Preissman [12] implies that there are no non-trivial compact products with negative sectional curvature. The following is one generalization:

Theorem 1.4. (1) *Let Σ be a connected, possibly infinite type oriented surface. There is a forward complete Finsler metric on $M = \Sigma \times T^n$ with negative flag curvature, if and only if Σ is the infinite cylinder or is $\mathbb{R}^2$.*

(2) *If Z is closed and admits a metric with finitely many class β geodesic strings (for some non-trivial class), then there is no Finsler metric on $M = Z \times T^n$ with a unique and non-degenerate geodesic string in class β.*

(3) *T^n does not admit a Finsler metric with a unique and non-degenerate geodesic string in some fixed non-trivial class β.*

When Σ is the infinite cylinder, a counterexample g to part one of the theorem, can be given as the warped product Riemannian metric.

Remark 1.5. The Riemannian version of the first part of the theorem above can be proved via the remarkable flat strip theorem [4]. Furthermore, the above results can be proved via S^1 equivariant Morse theory for the energy functional on the loop space of M, but the proofs are harder.

Part 1,2 of the above theorem are just a very special case of the following. We denote by $\pi_1^{inc}(X)$ the set of free homotopy classes of loops incompressible to the ends, in the sense of Definition 1.7, (non-constant classes when X is closed). Part 3 of the theorem above is proved.

Theorem 1.6. *Let $X = Z \times Y$ where Z, Y admit complete Riemannian metrics with non-positive sectional curvature, Y is closed, $\pi_1^{inc}(Z) \neq 0$, $\chi(Y) \neq \pm 1$. Then:*

- *X does not admit a complete metric of negative sectional curvature, or a forward complete Finsler metric with negative flag curvature.*
- *Moreover, X does not admit a forward complete Finsler metric with a unique and non-degenerate class β geodesic string for any $\beta \in \pi_1^{inc}(Z)$.*

This theorem is very close to being sharp, for example the conclusion of the corollary is false if $Y = S^1$ and $X = S^1 \times \mathbb{R}$. As $Z = T^2 \times \mathbb{R}$ admits the warped product metric $g_{T^2} \times_{e^t} g_{\mathbb{R}}$ (with respect to the function $f = e^t$ on $\mathbb{R}$) where $g_{T^2}, g_{\mathbb{R}}$ are the flat metrics. This warped product has constant negative sectional curvature -1. So it is essential that not only $\pi_1(Z) \neq 0$ but also that there is a class incompressible to the ends. Of course, $\chi(Y) = 1$ is also obviously essential, otherwise we may take $Y = pt$. The condition $\chi(Y) \neq -1$ is however not obviously essential.

We will also give various generalizations of this result to fibrations. For fibrations, the most obvious analogue of Preissman's theorem fails even assuming compactness. In fact, every closed 3-manifold X^3, for which there is no injection $\mathbb{Z}^2 \to \pi_1(X, x_0)$, and which fibers over a circle has a hyperbolic structure g_h, Thurston [16].

1.1. **Setup and more extensive statements.**

Terminology 1. From now on, all our metrics are Riemann-Finsler (a.k.a. Finsler) metrics unless specified to be Riemannian, and usually denoted by just g. Completeness, always means forward completeness, and it is an assumption for all our metrics. Curvature always means sectional curvature in the Riemannian case and flag curvature in the Finsler case. Thus we will usually just say complete metric g, for a forward complete Riemann-Finsler metric. A reader may certainly choose to interpret all metrics as Riemannian metrics, completeness as standard completeness, and curvature as sectional curvature.

In what follows $\pi_1(X)$ denotes the set of free homotopy classes of continuous maps $o : S^1 \to X$.

Definition 1.7. *Let X be a smooth manifold. Fix an exhaustion by nested compact sets $\bigcup_{i\in\mathbb{N}} K_i = X$, $K_i \supset K_{i-1}$ for all $i \geq 1$. We say that a class $\beta \in \pi_1(X)$ is end compressible if β is in the image of*

$$inc_* : \pi_1(X - K_i) \to \pi_1(X)$$

for all i, where $inc : X - K_i \to X$ is the inclusion map. We say that β is **end incompressible** *(or incompressible to the ends) if it is not end compressible.*

Let $\pi_1^{inc}(X)$ denote the set of such end incompressible classes. When X is compact, we set $\pi_1^{inc}(X) := \pi_1(X) - const$, where *const* denotes the set of homotopy classes of constant loops.

It is easily seen that the above is well defined (independent of the choice of an exhaustion) and moreover any homeomorphism $X_1 \to X_2$ of a pair of manifolds induces a set isomorphism $\pi_1^{inc}(X_1) \to \pi_1^{inc}(X_2)$. Denote by $L_\beta X$ the class $\beta \in \pi_1^{inc}(X)$ component of the free loop space of X, with its compact open topology. Let g be a complete metric on X, and let $S(g, \beta) \subset L_\beta X$ denote the subspace of all constant speed parametrized, smooth, closed g-geodesics in class β.

Definition 1.8. *We say that a metric g on X is* β**-taut** *if it is complete and $S(g, \beta)$ is compact. We will say that g is* **taut** *if it is β-taut for each $\beta \in \pi_1^{inc}(X)$.*

Lemma 1.9. *A complete metric g with non-positive curvature satisfies:*

- *All of its closed geodesics are minimizing in their free homotopy class.*
- *It is taut.*

Proof. The first part is a standard consequence of the Cartan-Hadamard theorem. The second part follows by the first part and Lemma 5.1. □

It should be emphasized that taut metrics form a much larger class of metrics then just non-positive curvature metrics. For example any sufficiently C^1 small perturbation of a metric with non-positive curvature will be taut. (Indeed, this is crucial for the construction of our invariant.) Another class of examples comes by way of Lemma 1.27 ahead, these metrics may not be non-positively curved nor nearby to metrics non-positively curved.

Definition 1.10. *Let $\beta \in \pi_1^{inc}(X)$, and let g_0, g_1 be a pair of β-taut metrics on X. A* **β-taut deformation** *between g_0, g_1, is a continuous (in the topology of C^0 convergence on compact sets) family $\{g_t\}$, $t \in [0,1]$ of complete metrics on X, s.t.*

$$S(\{g_t\}, \beta) := \{(o,t) \in L_\beta X \times [0,1] \,|\, o \in S(g_t, \beta)\}$$

is compact. We say that $\{g_t\}$ is a **taut deformation** *if it is β-taut for each $\beta \in \pi_1^{inc}(X)$. The above definitions of tautness are extended naturally to the case of a smooth fibration $X \hookrightarrow P \to [0,1]$, with a smooth fiber-wise family of metrics.*

A useful criterion for β-tautness is the following.

Theorem 1.11. *Let $\{g_t\}_{t\in[0,1]}$ be a continuous family of complete metrics on X. Suppose that:*

$$\sup_t |\max_{o \in S(g_t,\beta)} l_{g_t}(o) - \min_{o \in S(g_t,\beta)} l_{g_t}(o)| < \infty,$$

where l_{g_t} is the length functional with respect to g_t, then $\{g_t\}$ is β-taut. It follows that sky catastrophes of vector fields on closed manifolds are not geodesible by metrics all of whose geodesics are minimal, Appendix A.1.

For example, the hypothesis is trivially satisfied if g_t have the property that all their class β closed geodesics are minimal in their homotopy class.

Corollary 1.12. *If g_t, $t \in [0,1]$ have non-positive curvature then $\{g_t\}$ is taut.*

Proof. This follows by the theorem and by Lemma 1.9. □

Fuller at the end of [7] has asked for any metric conditions on vector fields to rule out sky catastrophes, see Appendix A.1. By the above, non-positivity of curvature is one such condition. So this is a partial answer to his question.

Remark 1.13. Note that if sky catastrophes were never geodesible (or at least if geodesible sky catastrophes are necessarily unstable, as was conjectured in [14]) then the geodesible Seifert conjecture would follow, by the main result of [14]. Hence, this is a subtle question. The qualitative structure of such potential geodesible or Reeb sky catastrophes is partially understood, [14, Theorem 1.10]. But this does not greatly aid constructing potential examples, which must be topologically very complex, (there are necessarily infinitely many suitably synchronized bifurcation events).

1.1.1. *The geodesic string counting invariant F.* Let $\mathcal{G}(X)$ be the set of equivalence classes of taut metrics g, where g_0 is equivalent to g_1 whenever there is a taut deformation between them. We may denote an equivalence class by its representative g by a slight abuse of notation.

Theorem 1.14. *For each manifold X there is a natural, non-trivial functional:*

$$\mathrm{F} : \mathcal{G}(X) \times \pi_1^{inc}(X) \to \mathbb{Q}.$$

The value $\mathrm{F}(g, \beta)$ is a certain weighted count of the set of closed g-geodesic strings in class β. But one must take care of exactly how to count, as in general this set should be understood as an orbifold or rather a Kuranishi space (as introduced by Fakaya-Ono [5]), hence this is why F is $\mathbb{Q}$ valued. In the special case when g is β-regular we have the following formula:

$$F(g, \beta) = \sum_{o \in \mathcal{O}(g,\beta)} \frac{(-1)^{\mathrm{morse}(o)}}{\mathrm{mult}(o)},$$

where morse(o) denotes the Morse index of o, (meaning the Morse-Bott index of the associated critical submanifold of the loop space), and mult(o) is as in the Introduction.

Corollary 1.15. *Suppose for a pair g_1, g_2 of β-taut metrics on X:*

$$F(g_1, \beta) \neq F(g_2, \beta),$$

then any path $\{g_t\}$, connecting g_0, g_1, is not β-taut and in fact has a sky catastrophe. So that if such a pair g_1, g_2 exists the conjecture of [14] would be disproved.

Proof. The fact that any connecting $\{g_t\}$ is not β-taut is just a direct corollary of the theorem above. The fact that $\{g_t\}$ has a sky catastrophe follows by [14, Theorem 3.2]. □

I was unable to find such a pair g_1, g_2, in fact there is strong evidence to believe the following:

Conjecture 1. *F does not depend on the choice of a smooth structure and taut metric. In other words we have the following. Let X be a topological manifold, define:*

$$\mathcal{S}(X) : \pi_1^{inc}(X) \to \mathbb{Q} \sqcup \{\infty\}$$

by:

$$\mathcal{S}(X)(\beta) = \begin{cases} \infty, & \text{if } X \text{ does not admit a smooth structure and a } \beta\text{-taut Finsler metric.} \\ F(g, \beta), & \text{if } g \text{ is a } \beta\text{-taut Finsler metric on } X. \end{cases},$$

then $\mathcal{S}$ is well defined and hence determines a topological invariant of topological manifolds. So if $f : X_1 \to X_2$ is a homeomorphism then $\mathcal{S}(X_1)(\beta) = \mathcal{S}(X_2)(f_*(\beta))$.

Theorem 1.20 in the following section partially proves this conjecture.

Remark 1.16. The smooth manifold version of this conjecture, i.e. that $\mathcal{S}(X)$ is a smooth manifold invariant, is implied by the dynamical conjecture of [14], see Remark 1.13. However, the above seems to be much more basic as will be apparent from the proof of Theorem 1.20.

Remark 1.17. The formulation of the conjecture naturally extends to Reeb vector fields. That is let λ be a contact form on the unit cotangent bundle C of a smooth manifold X (for simplicity closed), with λ representing the Louiville contact structure (i.e. the standard contact structure). Assume that the space of closed, class $\widetilde{\beta}$ λ-Reeb orbits on C is compact. Then the Fuller index (See Section 5) of this compact set is a topological invariant of X.

1.1.2. *Basic results on the invariant F.*

Definition 1.18. *Let $\beta \in \pi_1(X)$. For any based point $x_0 \in$ image $\beta \subset X$ (for image β the image of some representative of β) there is a naturally determined element $\beta_{x_0} \in \pi_1(X, x_0)$ well defined up to an inner automorphism, (concatenate a representative of β with a path from x_0 to a point in image β).*

- *We say that a class $\beta \in \pi_1(X, x_0)$ is* **not a power** *if whenever $\beta = \alpha^k$ for some $\alpha, k > 0$ then $k = 1$.*
- *We say that a class $\beta \in \pi_1(X, x_0)$ is a k-***power** *if $\beta = \alpha^k$, $k > 1$, for some α which is not a n-power for any n.*
- *We say that β is* **atomic** *if it is a k-power for some k.* [1]
- *We say that $\beta \in \pi_1(X)$ is not a power, respectively is a k-power, respectively is atomic if for any x_0 as above, β_{x_0} is not a power, respectively is a k-power, respectively is atomic.*

[1]As we are working with non-compact manifold, we may in principle have non atomic classes.

Note that if a class $\beta \in \pi_1^{inc}(X)$ is not a power then any representative of this class is not multiply covered, but the converse generally does not hold.

Example 1. Let g be a Riemannian metric with negative sectional curvature on a closed manifold X and $\beta \in \pi_1(X)$ a class represented by a multiplicity n closed geodesic, then

$$F(g, \beta) = \frac{1}{n}. \tag{1.19}$$

In particular, if β is not a power then $F(g, \beta) = 1$. More generally, (1.19) holds whenever g has a unique and non-degenerate geodesic string in class β, where non-degenerate is as in the Introduction.

If $\beta \in \pi_1^{inc}(X)$ is not a power, then it is easy to see that that the reparametrization S^1 action on $L_\beta X$ is free (see Appendix A), so that $H_*^{S^1}(L_\beta X, \mathbb{Z}) \simeq H_*(L_\beta X/S^1, \mathbb{Z})$, where $H_*^{S^1}(L_\beta X, \mathbb{Z})$ denotes the S^1-equivariant homology. Moreover, we have:

Theorem 1.20. *Suppose that $\beta \in \pi_1^{inc}(X)$ is not a power, and X admits a β-taut metric, then $H_*^{S^1}(L_\beta X, \mathbb{Z})$ is finite dimensional. Denote by $\chi^{S^1}(L_\beta X)$ the Euler characteristic of this homology. Then for any β-taut metric g on X:*

$$F(g, \beta) = \chi^{S^1}(L_\beta X).$$

In particular Conjecture 1 holds on the subset of classes β which are not a power.

Explicit examples for the theorem above can be found by the proof of Theorem 1.22. For these types of examples any negative integer may appear as the value of $F(g, \beta)$. We leave out the details.

Remark 1.21. If β is a power, the idea behind Theorem 1.20 breaks down, as the S^1-equivariant homology of $L_\beta X$ may then be infinite dimensional even if X admits a β-taut g. As a trivial example, this homology is already infinite dimensional when g is negatively curved, and the class β geodesic is k-covered, as then this homology is the group homology of $\mathbb{Z}_k$. However for a general β we should still have that $F(g, \beta)$ is the orbifold Euler characteristic, of a suitable orbifold quotient $L_\beta X/S^1$ which in principle proves Conjecture 1.

We can use the above, and the product formula of Theorem 11.1 to get:

Theorem 1.22. *Every rational number has the form $F(g, \beta)$ for some β-taut Riemannian g on some compact manifold X and for some β.*

1.1.3. *Applications to existence of negative curvature metrics.* A celebrated theorem of Preissman [12] says that there are no negative sectional curvature metrics on compact products. Fibration counterexamples to Preissman's product theorem certainly exist as mentioned in the Introduction. We are going to give a certain generalization of Preissman's theorem to fibrations, with possibly non-compact fibers, also replacing the negative sectional curvature condition by a significantly weaker condition.

Definition 1.23. *Let $Z \hookrightarrow X \xrightarrow{p} Y$ be a smooth fiber bundle with X having a β-taut Riemannian metric g, for $\beta \in \pi_1^{inc}(X)$, and let g_Y be a metric on Y. Suppose that:*

(1) *The fibers $Z_y = p^{-1}(y)$ are totally g-geodesic, for closed geodesics in class β. We denote by g_y the metric g restricted to Z_y.*
(2) *The fibers are parallel (the distribution $T^{vert}X = \ker p_*$ is parallel along any smooth curve in X with respect to the Levi-Civita connection of g).*
(3) *For any pair of fibers (Z_{y_0}, g_{y_0}), (Z_{y_1}, g_{y_1}), and a path $\gamma : [0,1] \to Y$ from y_0 to y_1 the fiber family $\{(Z_{\gamma(t)}, g_{\gamma(t)})\}$ furnishes a taut deformation.*

(4) *p projects g-geodesics to geodesics of Y, g_Y.*

We then call $p : X \to Y$ a **β-taut fibration***, with the metrics g, g_Y and g_Z all possibly implicit.*

Definition 1.24. *For $Z \hookrightarrow X \to Y$ as above, we say that $\beta \in \pi_1(X)$ is a* **fiber class** *if it is in the image of the inclusion $i_Z : \pi_1(Z) \to \pi_1(X)$.*

In the above definition of a taut fibration and the following theorem we need the auxiliary metric g on X to be Riemannian, and there is no obvious extension of the theorem to the Riemann-Finsler case. However, the conclusions of the theorem are for Riemann-Finsler metrics.

Theorem 1.25. *Let $p : (X, g) \to (Y, g_Y)$ be a β-taut fibration, where $\beta \in \pi_1^{inc}(X)$ is a fiber class. Suppose further that Y is connected, closed, $\chi(Y) \neq \pm 1$ and is such that all smooth closed contractible g_Y-geodesics in Y are constant. Then the following holds:*

- *X does not admit a complete Riemann-Finsler metric with negative curvature.*
- *Moreover, X does not admit a complete Riemann-Finsler metric with a unique and non-degenerate class β geodesic string.*

Note that $\chi(Y) \neq 1$ is of course essential, as the trivial fibration $X \to \{pt\}$, with X admitting a complete negatively curved metric, will satisfy the hypothesis. The condition that there is a fiber class $\beta \in \pi_1^{inc}(X)$ is also essential, for any vector bundle over a manifold admitting a Riemannian metric of negative curvature admits a metric of negative curvature, Anderson [1].

Theorem 1.6 gives one set of examples. A further basic set of examples for the theorem is obtained by starting with any homomorphism

$$\phi : \pi_1(Y, y_0) \to \mathrm{Isom}(Z, g_Z), \quad \text{(the group of all isometries).} \tag{1.26}$$

where g_Z is a taut Riemannian metric, and there is a class $\beta_Z \in \pi_1^{inc}(Z)$ (for example (Z, g_Z) is a non-simply connected complete hyperbolic surface). Suppose further:

(1) The orbit
$$O := \bigcup_{\gamma \in \pi_1(Y, y_0)} \phi_*(\gamma)(\beta_Z)$$
is finite.

(2) Y is closed and connected.

(3) All contractible smooth closed g_Y geodesics in Y are constant.

We have the obvious induced diagonal action

$$\pi_1(Y, y_0) \to \mathrm{Diff}(Z \times \widetilde{Y}), \text{ (the group of all diffeomorphisms),}$$

$$\gamma \mapsto ((z, y) \mapsto (\phi(\gamma)(z), \gamma \cdot y)),$$

for $\widetilde{Y}$ the universal cover of Y. Taking the quotient of $Z \times \widetilde{Y}$ by this action, we get an associated "flat" bundle $Z \hookrightarrow X_\phi \xrightarrow{p} Y$, with a metric g_ϕ induced from the product metric $\widetilde{g} = g_Z \oplus g_Y$, on the covering space $q : Z \times \widetilde{Y} \to Z \times Y$.

Lemma 1.27. *Let $p : (X_\phi, g_\phi) \to (Y, g_Y)$ be as above, then this is a β-taut fibration, where $\beta = i_*(\beta_Z)$, for $i_* : \pi_1^{inc}(Z) \to \pi_1^{inc}(X_\phi)$ induced by inclusion.*

By the lemma above, $p : (X_\phi, g_\phi) \to (Y, g_Y)$ satisfies the hypothesis of the theorem above. Yet more concretely:

Example 2. Suppose we have $\beta_Z \in \pi_1^{inc}(Z)$, and let $\phi : Z \to Z$ be an isometry of a taut metric g_Z. Then by the construction above, the mapping torus

$$(Z, g_Z) \hookrightarrow (X_\phi, g_\phi) \xrightarrow{\pi} S^1$$

has the structure of a β-taut fibration, satisfying the hypothesis of the theorem, for $\beta = i_*(\beta_Z)$ as above.

The next corollary of Theorem 1.25 is immediate.

Corollary 1.28. *Let*

$$(Z_{g,Z}) \hookrightarrow (X_\phi, g_\phi) \to (Y, g_Y)$$

be as in the construction above for Z, g_Z having non-positive curvature, and let $\beta_Z \in \pi_1^{inc}(Z)$. Then if $\chi(Y) \neq \pm 1$:

1. *X_ϕ does not admit a complete Riemann-Finsler metric with negative curvature.*
2. *Moreover, X_ϕ does not admit a Riemann-Finsler metric with a unique and non-degenerate class β geodesic string, for $\beta = i_*(\beta_Z)$ as above.*

As a special case, this applies to the mapping tori X_ϕ, for $\phi : Z \to Z$ an isometry of a complete Riemannian non-positively curved metric on Z, satisfying the finiteness condition 1. (The non-positive curvature hypothesis is for concreteness we may of course replace this condition by tautness.)

In the special case when Z is compact, the first part of the above corollary can be deduced, with some work, from Preissman's theorem (specifically, because of the condition 1), see also [3, Theorem 9.3.4] for a generalization that fits our Finsler setting. The second part is new even when Z is compact.

1.2. **Estimated counts of multiply covered geodesics.**

Theorem 1.29. *Suppose that g_1 is a taut metric on X, taut deformation equivalent to a complete metric of negative curvature (everything is Finsler). Suppose that $\beta \in \pi_1^{inc}(X)$ is a k-power. Let L_β be the length of a class β, g_1-geodesic. For all $\epsilon > 0$ sufficiently small, whenever g' is C^0 ϵ close to g_1, and is β-regular, we have:*

$$\sum_{o \in \mathcal{O}_{2L_\beta}(g', \beta)} \frac{(-1)^{\text{morse}(o)}}{\text{mult}(o)} = \frac{1}{k},$$

where $\mathcal{O}_{2L_\beta}(g', \beta)$ is the set of class β geodesic strings with g'-length less than $2L_\beta$.

2. Proof of Theorem 1.11

The first part of the theorem clearly follows by the second part. So let $\{g_t\}$, $t \in [0, 1]$ be as in the hypothesis, with

$$\sup_t |\max_{o \in S(g_t, \beta)} l_{g_t}(o) - \min_{o \in S(g_t, \beta)} l_{g_t}(o)| < C, \tag{2.1}$$

and suppose that

$$\sup_{(o,t) \in \mathcal{O}(\{g_t\}, \beta)} l_{g_t}(o) = \infty.$$

Then we have a sequence $\{o_k\}$, $k \in \mathbb{N}$, of closed class β g_{t_k}-geodesics in X, satisfying:

1. $\lim_{k \to \infty} t_k = t_\infty \in [0, 1]$.
2. $\lim_{k \to \infty} l_{g_{t_k}}(o_k) = \infty$, where $l_{g_{t_k}}(o_{t_k})$ is the length with respect to g_{t_k}.

Let o_∞ be a minimal, class β, $g_\infty = g_{t_\infty}$ geodesic in X. And let L denote its length g_∞ length. Let g_{aux} be a fixed auxiliary metric on X, and let L_{aux} be the g_{aux} length of o_∞.

Define a pseudo-metric d_{C_0} on the space of metrics on X as follows. Set $K = \operatorname{image} o_\infty$. And set

$$V \subset TX = \{v \in TX \mid \pi(v) \in K \text{ for } \pi : TX \to X \text{ the canonical projection, and } |v|_{aux} = 1\},$$

where $|v|_{aux}$ is the norm taken with respect to g_{aux}.

Then define:

$$d_{C^0}(g_1, g_2) = \sup_{v \in V} ||v|_{g_1} - |v|_{g_2}|.$$

By Properties 1 and 2 we may find a $k > 0$ such that:

$$d_{C^0}(g_{t_k}, g_{t_\infty}) < \epsilon \tag{2.2}$$

and

$$l_{g_{t_k}}(o_k) > C + L + L_{aux} \cdot \epsilon. \tag{2.3}$$

By (2.2), we have:

$$l_{g_{t_k}}(o_\infty) < l_{g_{t_\infty}}(o_\infty) + L_{aux} \cdot \epsilon = L + L_{aux} \cdot \epsilon.$$

Combining with (2.3) we get:

$$l_{g_{t_k}}(o_k) > l_{g_{t_k}}(o_\infty) + C.$$

Since we may find a closed g_{t_k}-geodesic o' satisfying $l_{g_{t_k}}(o') \leq l_{g_{t_k}}(o_\infty)$, we get that

$$|\max_{o \in S(g_{t_k}, \beta)} l_{g_{t_k}}(o) - \min_{o \in S(g_{t_k}, \beta)} l_{g_{t_k}}(o)| > C,$$

and so we are in contradiction.

Thus,

$$\sup_{(o,t) \in \mathcal{O}(\{g_t\}, \beta)} l_{g_t}(o) < \infty.$$

It follows, by an analogue of Lemma 5.2, that the images of all elements $o \in S(\{g_t\}, \beta)$ are contained in a fixed compact $T \subset X$. Compactness of $S(\{g_t\}, \beta)$ then readily follows by the Arzella-Ascolli theorem. □

3. Proof of Lemma 1.27

Let $\phi_* : \pi_1(Y, y_0) \to \operatorname{Aut}(\pi_1^{inc}(Z))$ be the natural induced action, where $\operatorname{Aut}(\pi_1^{inc}(Z))$ denotes the group of set isomorphisms of $\pi_1^{inc}(Z))$. And such that the orbit

$$O := \bigcup_{\gamma \in \pi_1(Y, y_0)} \phi_*(\gamma)(\beta_Z)$$

is finite.

As g_Z is taut, $S(g_Z, \phi_*(\gamma)(\beta_Z))$ is compact for each γ, where $S(g_Z, \phi_*(\gamma)(\beta_Z))$ is the space of geodesics as in Definition 1.8. By the condition on contractible geodesics of g_Y, we get:

$$\begin{aligned} S(g_\phi, \beta) &= q_*(S(g_Z \oplus g_Y, \beta))) \\ &= \bigcup_{\beta \in O} q_*(S(g_Z, \beta) \times \widetilde{Y}), \end{aligned}$$

for $q_* : L(Z \times \widetilde{Y}) \to L(Z \times Y)$ induced by the quotient map $q : Z \times \widetilde{Y} \to Z \times Y$, (as in the preamble to the statement of the lemma) and where $S(g_Z, \gamma) \times \widetilde{Y}$ is understood as a subset

$$S(g_Z, \gamma) \times \widetilde{Y} \subset L(Z) \times \widetilde{Y} \subset L(Z \times \widetilde{Y}).$$

Given that O is finite, this then readily implies our claim. □

4. Preliminaries on Reeb flow

Let (C^{2n+1}, λ) be a contact manifold with λ a contact form, that is a one form s.t. $\lambda \wedge (d\lambda)^n \neq 0$. Denote by R^λ the Reeb vector field satisfying:

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

Recall that a ***closed λ-Reeb orbit*** (or just Reeb orbit when λ is implicit) is a smooth map

$$o : (S^1 = \mathbb{R}/\mathbb{Z}) \to C$$

such that

$$\dot{o}(t) = cR^\lambda(o(t)),$$

with $\dot{o}(t)$ denoting the time derivative, for some $c > 0$ called period. Let $S(R^\lambda, \beta)$ denote the space of all closed λ-Reeb orbits in free homotopy class β, with its compact open topology. And set

$$\mathcal{O}(R^\lambda, \beta) = S(R^\lambda, \beta)/S^1,$$

where $S^1 = \mathbb{R}/\mathbb{Z}$ acts by reparametrization $t \cdot o(\tau) = o(t + \tau)$.

5. Definition of the functional F and proofs of auxiliary results

Let X be a manifold with a taut metric g. Let C be the unit cotangent bundle of X, with its Louiville contact 1-form λ_g. If $o : S^1 = \mathbb{R}/\mathbb{Z} \to X$ is a constant speed closed geodesic, it has a canonical lift $\widetilde{o} : S^1 \to C$, which is a closed flow line of $s \cdot R_{\lambda_g}$, where $s = |\frac{do}{dt}|$ is the speed of the geodesic, i.e. it is a Reeb orbit.

If $\beta \in \pi_1^{inc}(X)$, let $\widetilde{\beta} \in \pi_1(C)$ denote class $[\widetilde{o}] \in \pi_1(C)$, where o is a constant speed closed geodesic representing β.

Let $S(R^{\lambda_g}, \widetilde{\beta})$ be the orbit space as in Section 4, for the Reeb flow of the contact form λ_g. And set

$$\mathcal{O}_{g,\beta} = \mathcal{O}(R^{\lambda_g}, \widetilde{\beta}) := S(R^{\lambda_g}, \widetilde{\beta})/S^1,$$

which by construction is identified with the space of class β g-geodesic strings. By the tautness assumptions $\mathcal{O}_{g,\beta}$ is compact.

We then define

$$F(g, \beta) = i(\mathcal{O}_{g,\beta}, R^{\lambda_g}, \widetilde{\beta}) \in \mathbb{Q}$$

where the right hand side is the Fuller index as outlined in the Appendix A. As a basic example we have:

Lemma 5.1. *Suppose that g is a complete metric on X, all of whose class $\beta \in \pi_1^{inc}(X)$ geodesics are minimal, then g is β-taut.*

Proof. First we state a more basic lemma.

Lemma 5.2. *Suppose that g is a complete metric on X, $\beta \in \pi_1^{inc}(X)$ and let $S \subset L_\beta X$ be a subset on which the g-length functional is bounded from above. Then the images in X of elements of S are contained in a fixed compact subset of X.*

Proof. Suppose otherwise. Fix an exhaustion by nested compact sets

$$\bigcup_{i \in \mathbb{N}} K_i = X, \quad K_i \supset K_{i-1}.$$

Then either there is sequence $\{o_i\}_{i \in \mathbb{N}}$, $o_i \in S$ s.t. $o_i \in K_i^c$, for K_i^c the complement of K_i, which contradicts the fact that β is end incompressible. Or there is a sequence $\{o_k\}_{k \in \mathbb{N}}$, $o_k \in S$ s.t.:

(1) Each o_k intersects K_{i_0} for some i_0 fixed.

(2) For each $i \in \mathbb{N}$ there is a $k_i > i$ s.t. o_{k_i} is not contained in K_i.

Now if $\operatorname{diam}(o_k)$ is bounded in k, then condition 1 implies that o_k are contained in a set of bounded diameter. (Here $\operatorname{diam}(o_k)$ denotes the diameter of image o_k.) Consequently, by Hopf-Rinow theorem [2], o_k are contained in a compact set. But this contradicts condition 2, and the fact that K_i form an exhaustion of X.

Thus, we conclude that $\operatorname{diam}(o_k)$ is unbounded, but this contradicts the hypothesis. □

Returning to the proof of the main lemma. By assumption, closed, class $\beta \in \pi_1^{inc}(X)$ geodesics are g-minimizing in their homotopy class and in particular have fixed length. By the lemma above there is a fixed $K \subset X$ s.t. every class β closed geodesic has image contained in K. Then compactness of $S(g, \beta)$ follows by Arzella-Ascolli theorem.

□

Proof Theorem 1.14. Let $\beta \in \pi_1^{inc}(X)$, be given and let g be β-taut. We just need to prove that $F(g, \beta)$ is invariant under a β-taut deformation of g. So let $\{g_t\}$, $t \in [0,1]$ be a β-taut deformation of metrics on a compact manifold X. Let $R^{\lambda_{g_t}}$ be the geodesic flow on the g_t unit cotangent bundle C_t. Trivializing the family $\{C_t\}$ we get a family $\{R_t\}$ of flows on $C \simeq C_t$, with R_t conjugate to $R^{\lambda_{g_t}}$.

Let $\mathcal{O}(\{R_t\}, \widetilde{\beta})$ be the cobordism as in (A.2), where $\widetilde{\beta} \in \pi_1(C)$ is as above. Then $\mathcal{O}(\{R_t\}, \widetilde{\beta})$ is compact as $S(\{g_t\}, \beta)$ is compact by assumption.

Basic invariance of the Fuller index, that is (A.3), immediately yields: $F(g_0, \beta) = F(g_1, \beta)$. □

6. Proof of Theorem 1.29

We already know by Example 1 that $F(g_0, \beta) = \frac{1}{k}$ and hence by Theorem 1.14 $F(g_1, \beta) = \frac{1}{k}$. Let U denote the open subset of $L_\beta X$ consisting of loops with g_1-length less then $2L_\beta$. By [14, Lemma 4.1], for all $\epsilon > 0$ sufficiently small, for any g', C^0 ϵ close to g the following holds. Set $g'_t = (t-1) \cdot g_1 + t \cdot g'$, for $t \in [0,1]$, then

$$\widetilde{N} = \mathcal{O}(\{g'_t\}, \beta) \cap (U \times [0,1])$$

is an open and compact subset of $\mathcal{O}(\{g'_t\}, \beta)$.

Now set

$$N_1 = \widetilde{N} \cap (L_\beta X \times \{1\}),$$

and $N_0 = \mathcal{O}(g_1, \beta)$. By the invariance property (A.3) of the Fuller index, we then have that

$$\frac{1}{k} = i(N_0, R^{\lambda_{g_1}}) = i(N_1, R^{\lambda_{g'}}).$$

On the other hand, by construction and by index computations as in [14, Section 2]), we get:

$$i(N_1, R^{\lambda_{g'}}) = \sum_{o \in \mathcal{O}(g', \beta) \cap U} \frac{(-1)^{\mathrm{morse}(o)}}{\mathrm{mult}(o)}.$$

If ϵ is chosen to be sufficiently small then $\mathcal{O}(g', \beta) \cap U = \mathcal{O}_{2L_\beta}(g', \beta)$. So that we are done. □

7. Proof of Theorem 1.3

Let β be a non-zero class, as in the statement. Let $Y \subset T^n$ be a totally geodesic submanifold diffeomorphic to T^{n-1}, containing image β. Let $\alpha \in H^1(T^n, \mathbb{Z})$ be the Poincare dual of Y and let $p : T^n \to S^1$ be the classifying map of α. Then clearly p is a β-taut fibration with respect to g_0.

By Theorem 1.20, $F(g_0, \beta) = 0$. Analogously to the proof of Theorem 1.29, we get that for all $\epsilon > 0$ sufficiently small, for any g' as in the statement of our theorem we have:

$$\sum_{o \in \mathcal{O}_{2L_\beta}(g',\beta)} \frac{(-1)^{\text{morse}(o)}}{\text{mult}(o)} = 0.$$

But if ϵ is sufficiently small then $\mathcal{O}_{2L_\beta}(g', \beta) = \mathcal{O}(g', \beta)$ by the following. A g' geodesic string o has g_0 geodesic curvature approximately 0, then o must be approximately g_0 minimizing, i.e. approximately of length L_β (this is specific to the Euclidean metric g_0). Hence any g' geodesic string o can be assumed to have length less $2L_\beta$ by taking ϵ to be sufficiently small.

So we get:

$$\sum_{o \in \mathcal{O}(g',\beta)} \frac{(-1)^{\text{morse}(o)}}{\text{mult}(o)} = 0.$$

The conclusion follows by basic arithmetic.

□

8. Proof of Part 3 of Theorem 1.4

Suppose otherwise, and let g be such a metric, and let o be the unique and non-degenerate geodesic string in some class $\widetilde{\beta}$.

If $\widetilde{\beta}$ is not a power, $F(g, \widetilde{\beta})$ is g-independent by Theorem 1.20. So we get:

$$0 = F(g_0, \widetilde{\beta}) = F(g, \widetilde{\beta}) = 1,$$

where g_0 is as in the proof of Theorem 1.3 above. We have a contradiction, so that $\widetilde{\beta}$ is a k-power, and so that $\widetilde{\beta}_{x_0} = \beta^k$ for $k \geq 1$ and $\beta \in \pi_1(X, x_0)$, ($\widetilde{\beta}_{x_0}$ is as in Definition 1.18) and is not a power.

Let o be a class β geodesic string, then its k-cover is a class $\widetilde{\beta}$ geodesic string, and by the uniqueness assumption on the latter, o is likewise unique. Moreover, it is non-degenerate since its k-cover is non-degenerate. We may apply the argument above to get that β is a k'-power for some k'. But this is a contradiction. □

9. Proof of Theorem 1.1

Let $p : T^n \to S^1$ be the β-taut fibration with respect to g_0, as in the proof of Theorem 1.3 above. By Theorem 1.20, $F(g_0, \beta) = 0$. Hence, as we assumed Conjecture 1, for any other β-regular g on M with finitely β-class geodesic strings we have:

$$\sum_{o \in \mathcal{O}(g,\beta)} \frac{(-1)^{\text{morse}(o)}}{\text{mult}(o)} = 0.$$

The conclusion then readily follows by basic arithmetic. □

10. Proof of Theorem 1.20

This is an application of Morse theory. As g is β-taut, $S(g, \beta)$ is compact. Let

$$L = \sup_{o \in S(g,\beta)} \text{energy}_g(o),$$

where

$$\text{energy}_g : L_\beta X \to \mathbb{R},$$

is the function:

$$\text{energy}_g(o) = \int_{S^1} \langle \dot{o}(t), \dot{o}(t) \rangle_g dt. \tag{10.1}$$

Choose $C > L$ and let U denote the subspace of $L_\beta X$ consisting of loops with g-energy less than C. Now U has the homotopy type of $L_\beta X$. This can be proved without infinite dimensional Morse theory. We may use the finite dimensional broken geodesic approximation as in Milnor [10], (passing to the limit) and the fact that there are no geodesics in the complement of U.

If g' is sufficiently C^0 nearby to g and is β-regular than

$$F(g,\beta) = \sum_{o \in \mathcal{O}(g',\beta)\cap U} (-1)^{\mathrm{morse}(o)}.$$

The latter assertion is shown similarly to the proof of Theorem 1.29, except now there is no multiplicity weight since our geodesics are forced have multiplicity one, by the condition that β is not a power. To finish the proof we just need to show that $\sum_{o\in\mathcal{O}(g',\beta)\cap U}(-1)^{\mathrm{morse}(o)}$ is the Euler characteristic of U/S^1, since the latter is the Euler characteristic of $L_\beta X/S^1$.

Let us now denote by $\mathcal{L}_\beta X$ the Hilbert manifold of H^1 loops, in class β, as used for example in the classical work of Gromoll-Meyer [8]. We also denote by $\mathcal{U}$ the C-sublevel set analogous to U. The Hilbert manifold $\mathcal{L}_\beta X$ is well known to be homotopy equivalent to $L_\beta X$ with its previously used compact open topology.

The energy function $\mathrm{energy}_{g'} : \mathcal{L}_\beta X \to \mathbb{R}$, defined as above, is smooth, S^1 invariant and satisfies the Palais-Smale condition. The flow for its negative gradient vector field V is complete, and we can do Morse theory mostly as usual. This is understood starting with the work of Klingenberg [9], with the framework of Palais and Smale [11]. Note that all this also applies to $\mathcal{U}$. In our case, $\mathrm{energy}_{g'}$ is moreover a Morse-Bott function with critical manifolds C_o corresponding to S^1 families of geodesics, for each geodesic string o.

There is an induced Morse-Bott cell decomposition on $\mathcal{U}$, meaning a stratification formed by V unstable manifolds of the above mentioned critical manifolds C_o. This is Bott's extension of the fundamental Morse decomposition theorem. Now the S^1 action on $\mathcal{L}_\beta X$ is free by the condition that β is not a power. This action is continuous, so taking the topological S^1 quotient, we get a CW cell decomposition of $\mathcal{U}/S^1$, with one k-cell for each closed g'-geodesic string o in $\mathcal{U}$, with Morse index $\mathrm{morse}(o) = k$. (Here the Morse index is the Morse-Bott index of the critical manifold C_o.) All of the above is well understood, see for instance [8].

From the above cell decomposition, we readily get that the homology

$$H_*(\mathcal{U}/S^1,\mathbb{Z}) = H_*(U/S^1,\mathbb{Z}) = H_*(L_\beta X/S^1,\mathbb{Z}) = H_*^{S^1}(L_\beta X,\mathbb{Z})$$

is finite dimensional. And we get that:

$$\chi(U/S^1) = \sum_{o\in\mathcal{O}(g',\beta)\cap U} (-1)^{\mathrm{morse}(o)} \quad \text{(immediate from the cell decomposition).}$$

□

11. Proof of Theorem 1.25 and its corollaries

We first prove:

Theorem 11.1. *Let $p : X \to Y$ be a β-taut fibration as in the statement of Theorem 1.25 and $\beta \in \pi_1^{inc}(X)$ a fiber class. Then*

$$F(g,\beta) = card \cdot \chi(Y) \cdot F(g_Z,\beta_Z), \tag{11.2}$$

where $card \in \mathbb{N} - \{0\}$ is the cardinality of a certain orbit of the holonomy group (as explained in the proof), and where β_Z is as in Lemma 1.27.

Proof. We have a natural subset of $\mathcal{O}' \subset \mathcal{O}g, \beta$, consisting of all vertical geodesics, that is g-geodesics contained in fibers $p^{-1}(y) = Z_y$. In fact,

$$\mathcal{O}' = \mathcal{O}g, \beta, \tag{11.3}$$

for if o is any class β closed geodesic, the projection $p(o)$ is a contractible closed g_Y-geodesic, and by assumptions is constant.

In particular, there a natural continuous projection

$$\widetilde{p}: \mathcal{O}g, \beta \to Y, \quad \widetilde{p}(o) = y$$

where y is determined by the condition that

$$Z_y \supset \operatorname{image} o.$$

We will use this to construct a suitable (in a sense abstract i.e. not Reeb) perturbation of the vector field R^{λ_g}, using which we can calculate the invariant $F(g, \beta)$.

Fix a Morse function on f on Y, let $C = S^*X$ denote the g-unit cotangent bundle of X. For $v \in T_xX$ let $\langle v|$ denote the functional

$$T_xX \to \mathbb{R}, \quad w \mapsto \langle v, w\rangle_g.$$

Define $\widetilde{f}: C \to \mathbb{R}$ by

$$\widetilde{f}(\langle v|) := f(p(v)),$$

also define

$$P: C \to \mathbb{R}$$

by

$$P(\langle v|) := |P^{vert}(v)|^2_g,$$

where $P^{vert}(v)$ denotes the g-orthogonal projection of v onto the $T^{vert}_xX \subset T_xX$, for $T^{vert}X$ the vertical tangent bundle of X, i.e. the kernel of the bundle map $p_*: TX \to TY$.

Next define $F: C \to \mathbb{R}$ by:

$$F(\langle v|) := P(\langle v|) + \widetilde{f}(\langle v|).$$

Set

$$V_t = R^{\lambda_g} - t \operatorname{grad}_{g_S} F,$$

where the gradient is taken with respect to the Sasaki metric g_S on C [13] induced by g. The latter Sasaki metric is the natural metric for which we have an orthogonal splitting $TC = T^{vert}C \oplus T^{hor}C$, where $T^{vert}C$ is the kernel of $pr_*: TC \to TX$, induced by the natural projection $pr: C \to X$, and where $T^{hor}C$ is the g Levi-Civita horizontal sub-bundle.

Set $\mathcal{O}_t = \mathcal{O}(V_t, \widetilde{\beta})$, where $\widetilde{\beta}$ is as in Section 5.

Lemma 11.4. *We have:*

1. *For all $t \in [0, 1]$, $N_t := \mathcal{O}_t \cap \mathcal{O}_{g,\beta}$ is open and closed in $\mathcal{O}_t$.*
2. *For all $t \in (0, 1]$, $N_t = \cup_{y \in \operatorname{crit}(f)} \widetilde{p}^{-1}(y)$, where $\operatorname{crit}(f)$ is the set of critical points of f.*

Proof. It is easy to see that V_t is complete and without zeros. Suppose that $t > 0$. Let $\langle v_\tau|$, $\tau \in \mathbb{R}$ be the flow line of V_t, through $\langle v_0|$, i.e. $\langle v_\tau| = \phi_\tau(\langle v_0|)$, for ϕ_τ the time τ flow map of V_t. By the fact that the fibers of p are assumed to be parallel, we have that

$$R^{\lambda_g}(P) = 0, \quad \text{using the derivation notation.}$$

Also,

$$\operatorname{grad}_{g_S} \widetilde{f}(P) = 0,$$

which readily follows by the conjunction of g_S being Sasaki and the fibers of p being parallel. Consequently, the function

$$\tau \mapsto P(\langle v_\tau|) = |P^{vert}(v_\tau)|_g^2$$

is monotonically decreasing unless either:

(1) v_0 is tangent to $T^{vert}X$, in which case for all τ, v_τ are tangent to $T^{vert}X$ and $|P^{vert}(v_\tau)|_g^2 = 1$.

(2) For all τ, $|P^{vert}(v_\tau)|_g^2 = 0$.

In particular, the closed orbits of V_t split into two types.

(1) Closed orbits $o(\tau) = \langle v_\tau|$ with v_τ always tangent to $T^{vert}X$. In this case we may immediately, conclude that o is a lift to C of a closed g-geodesic contained in the fiber over a critical point of f.

(2) Closed orbits $o(\tau) = \langle v_\tau|$ for which v_τ is always g-orthogonal to $T^{vert}X$.

Clearly, the conclusion follows. □

Remark 11.5. It would be very fruitful to remove the condition on the fibers of p being parallel. But our argument would need to substantially change.

We return to the proof of the theorem. Set

$$\widetilde{N} = \{(o,t) \in L_{\widetilde{\beta}}C \times [0,\epsilon] \,|\, o \in N_t\},$$

where $L_{\widetilde{\beta}}C$ denotes the $\widetilde{\beta}$ component of the free loop space as previously. By part I of Lemma 11.4, this is an open compact subset of $\mathcal{O}(\{V_t\}, \widetilde{\beta})$ s.t.

$$\widetilde{N} \cap (L_{\widetilde{\beta}}C \times \{0\}) = \mathcal{O}(R^{\lambda_g}, \widetilde{\beta}),$$

(equalities throughout are up to natural set theoretic identifications.)

By definitions:

$$N_t = \widetilde{N} \cap (L_{\widetilde{\beta}}C \times \{t\}).$$

Now the basic invariance of the Fuller index, (A.3) gives:

$$i(N_0, R^{\lambda_g}, \widetilde{\beta}) = i(N_1, V_1, \widetilde{\beta}).$$

We proceed to compute the right hand side. Fix any smooth Ehresmann connection $\mathcal{A}$ on the fiber bundle $p: X \to Y$. This induces a holonomy homomorphism:

$$hol_y : \pi_1(Y,y) \to \operatorname{Aut}\pi_1(Z_y) \text{ (the right-hand side is the group of set automorphisms),}$$

with image denoted $\mathcal{H}_y \subset \operatorname{Aut}\pi_1(Z_y)$.

Let β_Z denote a class in $\pi_1(Z_y)$ s.t. $(i_{Z_y})_*(\beta_Z) = \beta$, for $i_{Z_y}: Z_y \to X$ the inclusion map. Set

$$S_y := \bigcup_{g \in \mathcal{H}_y} g(\beta_Z) \subset \pi_1(Z_y).$$

Then for another $y' \in Y$,

$$h_* : S_{y'} \to S_y, \tag{11.6}$$

is an isomorphism, where $h: Z_y \to Z_{y'}$ is a smooth map given by the $\mathcal{A}$-holonomy map determined by some path from y to y', and where h_* is the naturally induced map.

Denoting by g_y the restriction of g to the fiber Z_y, let R^y denote the λ_{g_y} Reeb vector field on the g_{Z_y}-unit cotangent bundle C_y of Z_y. The cardinality *card* of S_y is finite, as otherwise we get a contradiction to the compactness of $S(g, \beta)$. Now

$$\widetilde{p}^{-1}(y) = \bigcup_{\alpha \in S_y} \mathcal{O}(R^{\lambda_y}, \alpha).$$

From part 2 of Lemma 11.4 and by index computations as in [14, Section 2]), we get:

$$i(N_1, V_1, \widetilde{\beta}) = \sum_{y \in \operatorname{crit}(f)} (-1)^{\operatorname{morse}(y)} \cdot i(\widetilde{p}^{-1}(y), R^{\lambda_y}, \widetilde{\beta}),$$

where $\operatorname{morse}(y)$ denotes the Morse index of y. Now

$$\begin{aligned} i(\widetilde{p}^{-1}(y), R^{\lambda_y}, \widetilde{\beta}) &= \sum_{\alpha \in S_y} i(\mathcal{O}(R^{\lambda_y}), R^{\lambda_y}, \widetilde{\alpha}) \\ &= \sum_{\alpha \in S_y} F(g_y, \alpha). \\ &= card \cdot F(g_Z, \beta_Z), \end{aligned}$$

where the last equality follows by (11.6), and by the condition 3 in the Definition 1.8. And so the result follows. ☐

We return to the proof of Theorem 1.25. The first part immediately follows from the second, as any class $\beta \in \pi_1^{inc}(X)$ geodesic strings of a complete negatively curved Riemannian manifold X are unique. We prove second part. Suppose first that β is an n-power: $\beta = \alpha^n$, for some $n \geq 1$ where α is not a power. By the assumption that all contractible g_Y geodesics are constant, classical Morse theory Milnor [10] tells us that Y has vanishing higher homotopy groups $\pi_k(Y, y_0)$, $k \geq 2$. And in particular $i_{Z,*} : \pi_1(Z, p_0) \to \pi_1(X, p_0)$ is a group injection, by the long exact sequence of a fibration. It follows that $\alpha \in \pi_1^{inc}(X)$ is also a fiber class.

Now, any α-class g-geodesic string must be contained in a fiber of p. For otherwise we may find a β class g-geodesic string, which is not contained in a fiber of p, which would contradict (11.3). It readily follows that $p : X \to Y$ is also α-taut.

Now, if $\chi(Y) \neq \pm 1$ then by (11.2) $F(g, \alpha) \neq 1$, since $F(g_Z, \alpha_Z)$ is an integer by Theorem 1.20. By Theorem 1.20

$$F(g, \alpha) = \chi^{S^1}(L_\alpha X).$$

So if X admits a complete metric with a unique and non-degenerate class α g-geodesic string then we have:

$$F(g, \alpha) = \chi^{S^1}(L_\alpha X) = 1$$

(see Proof of Theorem 1.20), which is impossible. It follows that X does not admit a metric with a unique and non-degenerate class β g-geodesic string. For if o, o' are distinct, non-degenerate, class α g-geodesics strings, then the n-fold covers $o^n, (o')^n$ are class β, distinct non-degenerate g-geodesic strings.

We now prove the general case. Suppose by contradiction that X admits a metric with a unique and non-degenerate class β g-geodesic string o. By the above, β is not atomic. Now o covers a multiplicity one geodesic string $\widetilde{o}$ in some class $\widetilde{\beta} \in \pi_1^{inc}(X)$. Moreover, $\widetilde{o}$ is the unique geodesic string in its class, otherwise o would not be unique in its class. We prove that $\widetilde{\beta}$ is not a power, which will be a contradiction to β not being atomic and will complete the proof.

Suppose otherwise, so that $\widetilde{\beta}_{x_0} = \alpha^k$ for $k > 1$ and $\alpha \in \pi_1(X, x_0)$, (β_{x_0} is as in Definition 1.18). Let u be a class α, closed g-geodesic string (where α also denotes the class in $\pi_1^{inc}(X)$ corresponding to the based class α.) It is immediate that the k cover of u, u^k represents $\widetilde{\beta}$ and is a g-geodesic string. By the uniqueness, $\widetilde{o} = u^k$. But this contradicts simplicity of $\widetilde{o}$. So $\widetilde{\beta}$ is not a power.

□

Proof of Theorem 1.6. Let g_Z, g_Y be complete Riemannian metrics on Z respectively Y with non-positive curvature. Take the product metric $g = g_Z \times g_Y$ on $X = Z \times Y$. For a class $\beta \in \pi_1(Z)$ in the image of the inclusion $\pi_1(Z) \to \pi_1(X)$, the natural projection $X \to Y$ is automatically a β-taut fibration. Then the conclusion readily follows from Theorem 1.25.

□

Proof of Theorem 1.22. By Theorem 11.1 0 is certainly a value of the invariant F. We first prove that every negative rational number is the value of the invariant. Let p, q be positive integers. Let Y be a closed surface of genus $(p+1) > 1$ with a hyperbolic metric g_Y, let Z be the genus 2 closed surface with a hyperbolic metric g_Z and let $\beta_Z \in \pi_1^{inc}(Z)$ be the class represented by a $2 \cdot q$-fold covering of a simple closed loop representing a generator of the fundamental group of Z.

Let $X = Y \times Z$ with the product metric $g = g_Y \times g_Z$ and $p : X \to Y$ the canonical projection. By Theorem 11.1

$$F(g, \beta) = \chi(Y) \cdot F(g_Z, \beta_Z) = (-2p) \cdot \frac{1}{2q} = -\frac{p}{q},$$

where β is as in Lemma 1.27. So we proved our first claim.

Let again p, q be positive integers. Let Y be closed surface of genus 2, with a hyperbolic metric g_Y. And let Z be a manifold satisfying $F(g_Z, \beta_Z) = -\frac{p}{2q}$ for some β_Z-taut metric g_Z on Z and for some class $\beta_Z \in \pi_1^{inc}(Z)$. This exist by the discussion above. Let $g = g_Y \times g_Z$ be the product metric on $Y \times Z$, and β as above. Analogously to the discussion above we get:

$$F(g, \beta) = \chi(Y) \cdot F(g_Z, \beta_Z) = (-2) \cdot \frac{-p}{2q} = \frac{p}{q}.$$

□

A. Fuller index and sky catastrophes

Let X be a complete vector field without zeros on a manifold M. Set

$$S(X, \beta) = \{o \in L_\beta M \mid \exists p \in (0, \infty), \;\; o : \mathbb{R}/\mathbb{Z} \to M \text{ is a periodic orbit of } pX\}. \tag{A.1}$$

The above p is uniquely determined and we denote it by $p(o)$ called the period of o.

There is a natural S^1 reparametrization action on $S(X, \beta)$: $t \cdot o$ is the loop $t \cdot o(\tau) = o(t + \tau)$. The elements of $\mathcal{O}(X, \beta) := S(X, \beta)/S^1$ will be called ***orbit strings***. Slightly abusing notation we just write o for the equivalence class of o.

The multiplicity $m(o)$ of an orbit string is the ratio $p(o)/l$ for $l > 0$ the period of a simple orbit string covered by o.

We want a kind of fixed point index which counts orbit strings o with certain weights. Assume for simplicity that $N \subset \mathcal{O}(X, \beta)$ is finite. (Otherwise, for a general open compact $N \subset \mathcal{O}(X, \beta)$, we need to perturb.) Then to such an (N, X, β) Fuller associates an index:

$$i(N, X, \beta) = \sum_{o \in N} \frac{1}{m(o)} i(o),$$

where $i(o)$ is the fixed point index of the time $p(o)$ return map of the flow of X with respect to a local surface of section in M transverse to the image of o.

Fuller then shows that $i(N, X, \beta)$ has the following invariance property. For a continuous homotopy $\{X_t\}$, $t \in [0,1]$ set

$$S(\{X_t\}, \beta) = \{(o,t) \in L_\beta M \times [0,1] \,|\, o \in S(X_t)\}.$$

And given a continuous homotopy $\{X_t\}$, $X_0 = X$, $t \in [0,1]$, suppose that $\widetilde{N}$ is an open compact subset of

$$\mathcal{O}(\{X_t\}, \beta) := S(\{X_t\}, \beta)/S^1, \tag{A.2}$$

such that

$$\widetilde{N} \cap (L_\beta M \times \{0\})\,/S^1 = N.$$

Then if

$$N_1 = \widetilde{N} \cap (L_\beta M \times \{1\})\,/S^1$$

we have

$$i(N, X, \beta) = i(N_1, X_1, \beta). \tag{A.3}$$

We call this ***basic invariance***. In the case $\mathcal{O}(X_0, \beta)$ is compact, $\mathcal{O}(X_1, \beta)$ is compact for any sufficiently C^0 nearby X_1, and in this case basic invariance implies (see for instance [14, Proof of Lemma 1.6]):

$$i(\mathcal{O}(X_0, \beta), X, \beta) = i(\mathcal{O}(X_1, \beta), X_1, \beta). \tag{A.4}$$

A.1. **Blue sky catastrophes.**

Definition A.5 (Preliminary)**.** *A* **sky catastrophe** *for a smooth family $\{X_t\}$, $t \in [0,1]$, of non-vanishing vector fields on a closed manifold M is a continuous family of closed orbit strings $\tau \mapsto o_{t_\tau}$, o_{t_τ} is an orbit string of X_{t_τ}, $\tau \in [0, \infty)$, such that the period of o_{t_τ} is unbounded from above.*

A sky catastrophe as above was initially constructed by Fuller [6]. Or rather his construction essentially contained this phenomenon. A more general definition appears in [14], we slightly extend it here to the case of non-compact manifolds. All these definitions become equivalent given certain regularity conditions on the family $\{X_t\}$ and assuming M is compact.

Definition A.6. *Let $\{X_t\}$, $t \in [0,1]$ be a continuous family of non-zero, complete smooth vector fields on a manifold M and $\beta \in \pi_1^{inc}(X)$.*

We say that $\{X_t\}$ has a **catastrophe in class** *β, if there is an element*

$$y \in \mathcal{O}(X_0, \beta) \sqcup \mathcal{O}(X_1, \beta) \subset \mathcal{O}(\{X_t\}, \beta)$$

such that there is no open compact subset of $\mathcal{O}(\{X_t\}, \beta)$ containing y.

A vector field X on M is ***geodesible*** if there exists a metric g on M s.t. every flow line of X is a unit speed g-geodesic. A family $\{X_t\}$ is ***geodesible*** if there is a continuous family $\{g_t\}$ of metrics, with X_t geodesible with respect to g_t for each t. A family $\{X_t\}$ is ***geodesible*** if there is a continuous family $\{g_t\}$ of metrics with X_t geodesible with respect to g_t for each t. A ***geodesible sky catastrophe*** is a geodesible family $\{X_t\}$ with a sky catastrophe. A ***Reeb sky catastrophe*** is a family of Reeb vector fields $\{X_t\}$ with a sky catastrophe.

12. Acknowledgements

I am grateful to the IAS for resources provided during my stay as a member, where some of this paper was written.

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