HAMILTONIAN ELEMENTS IN ALGEBRAIC K-THEORY

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Abstract. Recall that topological complex K-theory associates to an isomorphism class of a complex vector bundle E over a space X an element of the complex K-theory group of X. Or from algebraic K-theory perspective, one assigns a homotopy class $[X \to K(\mathcal{K})]$, where $\mathcal{K}$ is the ring of compact operators on the Hilbert space. We show that there is an analogous story for algebraic K-theory of a general commutative ring k, replacing, and in a sense generalizing complex vector bundles by certain monotone/Calabi-Yau Hamiltonian fiber bundles. (In Calabi-Yau setting k must be restricted.) In suitable cases, we may first assign elements in a certain categorified algebraic K-theory, analogous to Toën's secondary K-theory of k. And there is a natural "Hochschild" map from this categorified algebraic K-theory to the classical variant. In particular, if k is regular and G is a compact Lie group we obtain a natural group homomorphism $\pi_m(BG) \to K_m(k) \oplus K_{m-1}(k)$. This story leads us to formulate a generalization of the homological mirror symmetry phenomenon to the algebraic K-theory context, based on ideas of gauged mirror symmetry of Teleman, and the formalism of Langlands dual groups.

1. Introduction

Our main goal is to tie up a pair of at a glance unrelated threads in geometry and in algebraic topology/algebra. The first thread is the existence of a homotopy coherent action, in the sense of Porter-Stasheff [11], of the group of Hamiltonian symplectomorphisms on the Fukaya category, particularly as developed in the author's [15] and [13], see also Oh-Tanaka [9] for a different perspective. The second thread is ideas of Toën [23] on secondary K-theory of commutative rings k, roughly speaking replacing k-modules by dg-categories over k, and using Waldhausen K-theory. This culminates in a connection of symplectic geometry and algebraic K-theory, and in particular we get a certain geometric insight into what we call categorified algebraic K-theory of $\mathbb{Z}$. We are then naturally lead to speculate on generalizations of the homological mirror symmetry phenomenon to algebraic K-theory context. We now describe some of the ingredients.

Although in the main examples here the principal actors are compact groups and $PU(n)$ in particular, the framework is most naturally understood from the perspective of the group of Hamiltonian symplectomorphisms of some symplectic manifolds, so we quickly introduce this. Let (M, ω) be a symplectic manifold, throughout assumed to be compact, with ω a closed non-degenerated 2-form on M. The group of diffeomorphisms ϕ of M, satisfying $\phi^*\omega = \omega$, is called the symplectomorphism group of (M, ω). When M is simply connected, this is also the group of Hamiltonian symplectomorphisms that we denote by $\mathcal{H} = \mathrm{Ham}(M, \omega)$. In general, a symplectomorphism is Hamiltonian if it is the time one map of an isotopy generated by a family of vector fields $\{X_t\}$, $t \in [0, 1]$, satisfying:

$$\omega(X_t, \cdot) = dH_t(\cdot),$$

for some family of smooth functions $\{H_t\}$ on M.

The group $\mathcal{H}$ is one of the more enigmatic objects of mathematics. It is always (unless $M = pt$) an infinite dimensional Fréchet Lie group with its C^∞ topology. But it has a natural, bi-invariant Finsler metric called the Hofer metric. And so in some ways it behaves like a compact group. It's topological and algebraic structure is tied to the main problems of Hamiltonian dynamics, see for instance Polterovich [10].

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Much of the algebraic/topological structure of $\mathcal{H}$ is reflected in the properties of Hamiltonian fibrations, which are the main geometric ingredients in what follows. A *Hamiltonian fibration* $M \hookrightarrow P \to X$, is a fiber bundle over X, with fiber a symplectic manifold (M, ω), whose structure group is contained in $\mathrm{Ham}(M, \omega)$. For example, take $(M, \omega) = (\mathbb{CP}^n, \omega_{FS})$ where ω_{FS} is the Fubini-Study symplectic form, and take the structure group $G = PU(n+1) \subset \mathrm{Ham}(\mathbb{CP}^n, \omega_{FS})$.

The other ingredient is algebraic K-theory, which is a construction associating an infinite loop space $K(k)$ [1] to any ring k, and so abelian groups:

$$K_m(k) = \pi_m(K(k)), \quad m \in \mathbb{N},$$

called K-theory groups of k. This is a powerful tool in algebraic topology and ring theory, with applications in number theory, geometry and geometric topology.

Taking k to be the ring $\mathcal{K}$ of compact operators on a separable complex Hilbert space, the algebraic K-theory space $K(\mathcal{K})$ [2] is identified with the topological K-theory space $KU = BU \times \mathbb{Z}$, see Karoubi [4], Suslin-Wodzicki [19]. In particular, a complex vector bundle E over a space X determines a homotopy class $[f_E : X \to K(\mathcal{K})]$, which we might call a *geometric element* of the corresponding abelian group. One goal is to extend this to a general commutative k.

There are two versions of the construction, corresponding to the pair of geometric cases of monotone and Calabi-Yau symplectic manifolds. In the monotone case, for each commutative ring k, we construct an infinite loop space $K^{Cat,\mathbb{Z}_2}(k)$. This is obtained by Waldhausen K-theory construction on a certain category, whose objects are pretriangulated A_∞ categories over k, with a certain algebraic finiteness condition that we call finite monotone type, Definition 5.16. This categorified algebraic K-theory of k is a mostly minor variation of the secondary K-theory of k introduced by Toën. $K^{Cat,\mathbb{Z}_2}(k)$ admits a natural map to the 2-periodic variant $K^{\mathbb{Z}_2}(k)$ of algebraic K-theory of k. Technically, $K^{\mathbb{Z}_2}(k)$ is the algebraic K-theory built using the exact category of 2-periodic complexes of k-modules.

Remark 1.1. It was pointed out to me by Bertrand Toën that for k a regular ring, for example $k = \mathbb{Z}$, $K^{\mathbb{Z}_2}_m(k) \simeq K_m(k) \oplus K_{m-1}(k)$. This can be proved with A^1 homotopy theory.

To obtain "geometric elements", we start with a Hamiltonian fibration $M \hookrightarrow P \to X$, with fiber a finite monotone type symplectic manifold. We show that for each k, the data of the isomorphism class of P, determines a homotopy class $[f_P : X \to K^{Cat,\mathbb{Z}_2}(k)]$. As one consequence of this, we get:

Theorem 1.2. *There is a natural homotopy class* $[g : BPU(n) \to K^{Cat,\mathbb{Z}_2}]$ *and so natural group homomorphisms:*

$$\pi_m(\mathrm{BPU}(n)) \xrightarrow{g_*} K^{Cat,\mathbb{Z}_2}_m(k) \to K^{\mathbb{Z}_2}_m(k), \quad \forall m, n \in \mathbb{N}, \tag{1.3}$$

for each commutative ring k. Furthermore, for any compact Lie group G, there is a homotopy natural map:

$$h : \mathrm{BG} \to K^{\mathbb{Z}_2}(k). \tag{1.4}$$

Example 1. In the case of $PU(2)$, in [13] we outline the proof injectivity of g_* for $m = 4$ and $k = \mathbb{Z}, \mathbb{Q}$. We discuss an extension of this in Section 6. Injectivity for $m = 2$ will be shown elsewhere. Also, $K_1(\mathbb{Z}) = \mathbb{Z}_2$, $K_2(\mathbb{Z}) = \mathbb{Z}_2$, so assuming the Remark 1.1, we may expect that the induced map

$$h_* : \mathbb{Z}_2 \simeq \pi_2(\mathrm{BPU}(2)) \to K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z}),$$

is injective. Also $K_3(\mathbb{Z}) = \mathbb{Z}_{48}$ and $K_4(\mathbb{Z}) = 0$. So given that g_* is injective for $m = 4$, $k = \mathbb{Z}$ we must conclude that the natural map $K^{Cat,\mathbb{Z}_2}_4(\mathbb{Z}) \to K^{\mathbb{Z}_2}_4(\mathbb{Z})$ has a kernel.

[1]In this paper, the term infinite loop space is meant to be synonymous with the term Ω-spectrum (not necessarily connective), i.e. a sequence of spaces $\{E_n\}_{n \in \mathbb{N}}$ with $E_n \simeq \Omega E_{n+1}$. Then $K(k)$ is the notation for the 0-space E_0 of the corresponding spectrum.

[2]Rather one must take a certain non-connective variant of the algebraic K theory construction.

Question 1. Can one obtain the classes

$$[\mathrm{BG} \to K^{\mathbb{Z}_2}(k)]$$

by homotopy theory techniques, perhaps aided by some algebraic geometry? In other words, can we bypass the geometric analysis of the construction?

A similar construction also exists for Hamiltonian fibrations with finite Calabi-Yau type fiber. In this case we must restrict k, necessitated by the condition that the Fukaya category is suitably well defined with k coefficients. Usually one takes k to be a certain Novikov field Λ over $\mathbb{C}$. In particular, a Hamiltonian fibration with such a finite Calabi-Yau type fiber over S^n, induces an element of the standard algebraic K-theory group $K_n(\Lambda)$.

2. Preliminaries

Let k be a commutative ring. Denote by $A_\infty Cat_k$ the category of strictly unital, $\mathbb{Z}$ graded A_∞ categories with morphisms A_∞ functors. We set $A_\infty Cat_k^{\mathbb{Z}_n}$ to be like $A_\infty Cat_k$ but with chain complexes $\mathbb{Z}_n$ graded instead of $\mathbb{Z}$ graded. This means that all structure maps, functors, etc., are with respect the $\mathbb{Z}_n$ grading. We will assume homological grading conventions on all chain complexes. Various subscripts k may be omitted.

There is an endofunctor

$$Perf : A_\infty Cat \to A_\infty Cat,$$

s.t. $Perf(B)$ is idempotent complete and pretriangulated for each B. One reference for this material is [2, Section 2.2]. We may denote by $\widehat{C}$ the category $Perf(C)$.

Let Cat_k denote the category of small k-linear categories. For $C \in \mathcal{A}, \mathcal{A}^{\mathbb{Z}_n}$, we set $hC \in Cat_k$ to be the category with the same objects and with $hom_{hC}(a, b) = H_0(hom_C(a, b))$.

Given a morphism $f : A \to B$ in $\mathcal{A}, \mathcal{A}^{\mathbb{Z}_n}$, we denote by $f_* : hA \to hB$ the induced morphism in Cat_k. We say that that f is *quasi fully-faithful* if f_* is fully-faithful, and f is a *quasi equivalence* if f_* is an equivalence. We will say that f is a *Morita equivalence* if $Perf(f) : Perf(A) \to Perf(B)$ is a quasi equivalence.

3. Building a Waldhausen category with objects A_∞ categories

To build an algebraic K-theory from the category of A_∞ categories, we need some analogue of "exact sequences" of A_∞ categories. For *dg* categories there are already very good answers to this, and this results in a definition of "secondary K-theory", Toën [22]. The main difficulty in the case of A_∞ categories (with non-strict functors) is the existence of suitable colimits. In fact, strict colimits of diagrams in $A_\infty Cat$ do not exist. This is not surprising because the "correct" (in some geometric and algebraic sense, for instance Keller [5]) notion of functors of A_∞ categories, preserves A_∞ structures only in homotopy coherent sense, the latter is also discussed in Porter-Stasheff [11]. Despite this, there is a suitable theory of homotopy colimits, which will be sufficient for our purposes.

We also need to put some algebraic finiteness conditions on our A_∞ categories, otherwise any K-theory we construct will be trivial due to the Eilenberg swindle, the same reason that K-theory of infinite dimensional vector bundles is trivial. One natural idea is to require our A_∞ categories be smooth and proper or saturated in the sense of Kontsevich-Soibelman [7] , as this is a dualizability condition, see [23]. We will work with a somewhat less restrictive notion, which naturally generalizes to the $\mathbb{Z}_n$ graded case, and is not too hard to check for geometric examples. Correspondingly, we will get a pair of categories $\mathcal{A}_k$ and $\mathcal{A}_k^{\mathbb{Z}_n}$, with the latter corresponding to the n-periodic case.

Let $\mathcal{A}_k \subset A_\infty Cat_k$ be the full-subcategory whose objects are pretriangulated, A_∞ categories B s.t. the following holds:

Condition 3.1. *B is Morita equivalent to an A_∞ algebra, whose underlying chain complex is perfect, and whose Hochschild homology complex is perfect.*

Example 2. Let C be smooth and proper dg category, then it satisfies Condition 3.1, Toën-Vaquié [24].

The category $\mathcal{A}_k^{\mathbb{Z}_n} \subset A_\infty Cat_k^{\mathbb{Z}_n}$ is defined to be the full-subcategory whose objects are pretriangulated, A_∞ categories B s.t. the following holds:

Condition 3.2. *B is Morita equivalent to an A_∞ algebra, whose underlying chain complex $B_\bullet$, in each degree is projective and finitely generated over k. The same holds for the Hochschild homology complex of B.*

It is shown in [3, Section A.4] that $\mathcal{A}$, $\mathcal{A}^{\mathbb{Z}_n}$ have certain homotopy colimits, in particular for the pushout diagram:

$$\bullet \longleftarrow \bullet \longrightarrow \bullet. \tag{3.3}$$

Remark 3.4. Strictly speaking only distinguished homotopy cocones are constructed in [3], although they are expected to be homotopy colimits. Moreover, these distinguished cocones are suitably functorial, that is maps of diagrams induces natural maps of these distinguished cocones. The structure of distinguished functorial homotopy cocones is enough for the S-construction underlying the infinite loop space (4.1). This is a very minor generalization of the standard Waldhausen S-construction. To simplify the discussion, we will just call them homotopy colimits.

To define a Waldhausen structure on $\mathcal{A}, \mathcal{A}^{\mathbb{Z}_n}$ we need to specify a class of morphisms $\mathcal{C}$ called cofibrations and a class of morphisms $\mathcal{W}$ called weak equivalences. We set these to be quasi fully-faithful maps and quasi isomorphisms respectively.

Lemma 3.5. *With respect to $\mathcal{C}, \mathcal{W}$ as above, both $\mathcal{A}$ and $\mathcal{A}^{\mathbb{Z}_n}$ are Waldhausen categories. More specifically, we have:*

1. *$\mathcal{A}, \mathcal{A}^{\mathbb{Z}_n}$ have a zero object.*
2. *The canonical morphism $0 \to A$ is a cofibration for all $A \in \operatorname{obj} \mathcal{A}, \operatorname{obj} \mathcal{A}^{\mathbb{Z}_n}$.*
3. *All isomorphisms are weak equivalences, and all weak equivalences are cofibrations.*
4. *Weak equivalences are closed under composition.*
5. *All diagrams of the form (3.3), where the right pointing arrow is a cofibration, have homotopy colimits.*
6. *Given a homotopy push-out diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] & D, \end{tikzcd}$$

   if f is a cofibration then so is g.

7. *Given a diagram:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, "e"] \arrow[d] & C \arrow[d] \\ D \arrow[r, "g"] & E \arrow[r, "e'"] & F, \end{tikzcd}$$

   where e, e' are cofibrations, and vertical arrows are weak equivalences, the induced map of homotopy push-outs is a weak equivalence.

Proof. The zero object is defined to be the zero category (the category with unique object and morphism). Property 2 then holds vacuously. Property 3 and 4 hold by definitions of $\mathcal{C}, \mathcal{W}$. To check property 6 we can work directly with the construction in [3, Appendix A.4].

Specifically, if

$$F : J \to A_\infty Cat$$

is the functor representing the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ C & \end{tikzcd},$$

then first we form a type of Grothendieck construction $\mathrm{Groth}\, F$, which is an A_∞ category, which we briefly describe. There are fully-faithful embeddings $A, B, C \to \mathrm{Groth}\, F$, with images denoted by just A, B, C, s.t.

$$\mathrm{obj}\, \mathrm{Groth}\, F = \mathrm{obj}\, A \sqcup \mathrm{obj}\, B \sqcup \mathrm{obj}\, C.$$

There is an identification

$$hom_{\mathrm{Groth}\, F}(a, b) = hom_B(f(a), b),$$

for $a \in A, b \in B$. And a similar identification for $hom_{\mathrm{Groth}\, F}(a, c)$. We call morphisms corresponding to the identity morphisms in this identification, *adjacent identities*.

These are all the non-zero hom sets, i.e. there is a non-zero morphism from $o_1 \in F(i_1)$ to $o_2 \in F(i_2)$ only if there is a morphism from i_1 to i_2 in J.

The homotopy colimit D is then defined to be the localization (an A_∞ analogue of the usual localization, as defined in [3]) of $\mathrm{Groth}\, F$ at the set of adjacent identity morphisms. There is then a homotopy commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] & D. \end{tikzcd}$$

By construction, the induced commutative diagram:

$$\begin{tikzcd} hA \arrow[r, "f_*"] \arrow[d] & hB \arrow[d] \\ hC \arrow[r, "g_*"] & hD, \end{tikzcd}$$

is a push out diagram in Cat_k. As f_* is fully-faithful, g_* is fully-faithful by Stanculescu [18, Proposition 3.1].

The final property is verified in [3]. $\square$

4. Categorified algebraic K-theory

We define $K^{Cat}(k)$ to be the infinite loop space given by the Waldhausen S-construction [25], on the Waldhausen category $(\mathcal{A}_k, \mathcal{W}, \mathcal{C})$.

More specifically,

$$K^{Cat}(k) = \Omega |wS_\bullet(\mathcal{A}_k)|, \tag{4.1}$$

where (very briefly)

- $S_\bullet(\mathcal{A}_k)$ is a simplicial category. For each d, objects of $S_\bullet(\mathcal{A}_k)(d)$ are certain length d sequences of cofibrations in $\mathcal{A}_k$ (with an additional structure.)

- $wS_\bullet(\mathcal{A}_k)$ is the simplicial category with $wS_\bullet(\mathcal{A}_k)(d) = w(S_\bullet(\mathcal{A}_k)(d))$, where the right-hand side is the category of weak equivalences of the category $S_\bullet(\mathcal{A}_k)(d)$.
- $|\cdot|$ is the geometric realization of a simplicial category, with the latter understood as a bi-simplicial set, taking nerve of each $wS_\bullet(\mathcal{A}_k)(d)$.

Likewise, $K^{Cat,\mathbb{Z}_n}(k)$ is defined by the S-construction on $(\mathcal{A}_k^{\mathbb{Z}_n}, \mathcal{W}, \mathcal{C})$.

4.1. **Some computation, and relation with Toën's secondary K-theory.** Toën [23], constructs what he calls the secondary K-theory spectrum $K^{(2)}(k)$ of k. Our construction above is an essentially minor modification of this construction. More specifically, Toën works with smooth and proper $\mathbb{Z}$-graded, pretriangulated *dg* categories. As mentioned, for *dg* categories the smooth and proper condition in particular implies the finiteness Condition (3.1), but can only be expected to hold for Fukaya categories of compact Calabi-Yau manifolds. Nevertheless, the spectrum $K^{(2)}(k)$ should be closely related to the spectra $K^{Cat,\mathbb{Z}_n}(k)$ and $K^{Cat}(k)$.

Define:

$$K_m^{Cat,\mathbb{Z}^n}(k) = \pi_m(K^{Cat,\mathbb{Z}^n}(k)). \tag{4.2}$$

And define:

$$K_m^{Cat}(k) = \pi_m(K^{Cat}(k)). \tag{4.3}$$

To partially demystify this, we note that $K_0^{Cat}(k)$ has the following description (similarly with $K_0^{Cat,\mathbb{Z}_n}(k)$). By a cofiber sequence of A_∞ functors we mean a diagram in $\mathcal{A}_k$:

$$A \xrightarrow{f} B \to C, \tag{4.4}$$

with f quasi fully-faithful, which completes to a homotopy pushout diagram in $\mathcal{A}_k$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C. \end{tikzcd}$$

Then $K_0^{Cat}(k)$ is the quotient of the free abelian group, generated by the set of isomorphism classes of objects in $\mathcal{A}_k$, by the subgroup generated by

$$\{[B] - [A] - [C] \,|\, \text{there is a cofiber sequence (4.4).}\}$$

Let us also recall the definition of $K_0(k)$. This is the quotient of the free abelian group, generated by the set of isomorphism classes of projective finitely generated k modules, by the subgroup generated by

$$\{[B] - [A] - [C] \,|\, \text{exists an exact sequence } 0 \to B \to A \to C \to 0.\}$$

More generally if $\mathcal{E}$ is an exact category, one defines $K_0(\mathcal{E})$ to be the quotient of the free abelian group, generated by the set of isomorphism classes in $\mathcal{E}$, by the subgroup generated by

$$\{[B] - [A] - [C] \,|\, \text{exists an exact sequence } B \to A \to C.\}$$

It is well known that $K_0(D^{perf}(Mod_k))$ is isomorphic to $K_0(k)$, (together with higher K-groups), where Mod_k denotes the category of projective finitely generated k modules and $D^{perf}(Mod_k)$ denotes the derived category of perfect complexes.

Lemma 4.5 (Toën [23])**.** *Denote by $Hoch_\bullet(C)$ the Hochschild homology complex of $C \in \mathcal{A}_k$. There is a natural map*

$$\begin{aligned} K_0^{Cat}(k) &\to K_0(D^{perf}(Mod_k)) \simeq K_0(k) \\ [C] &\mapsto [Hoch_\bullet(C)], \end{aligned}$$

and it is non-trivial, where $[\cdot]$ denote the classes in the associated Grothendieck K-groups.

Toën's proof of this proceeds as follows. One shows that images of cofiber sequences of A_∞ functors, by the functor $C \mapsto Hoch_\bullet(C)$, give exact sequences in $D^{perf}(Mod_k)$, of the associated Hochschild complexes. This property of the Hochschild functor is discussed for example in Keller [5, Theorem 5.2]. Other interesting computations in degree 0 have been made by Tabuada [20].

There is an analogous map in the case of $K_0^{Cat,\mathbb{Z}_n}(k)$ but in general we no longer have that $K_0(k) \simeq K_0(D_n(Mod_k))$, where $D_n(Mod_k)$ denotes the n-periodic derived category, as there are certain corrections, see Saito [12]. We do have:

Theorem 4.6 (Saito [12])**.** *For n even, the natural homomorphism:*

$$\psi : K_0(D_n(Mod_k)) \to K_0(k) \tag{4.7}$$

$$\psi([V]) = \sum_{i=0}^{i=n} (-1)^i [H_i(V)], \tag{4.8}$$

is an isomorphism.

Corollary 4.9. *The composition map*

$$K_0^{Cat,\mathbb{Z}_2}(\mathbb{Z}) \xrightarrow{\phi} K_0(D_2(Mod_{\mathbb{Z}})) \xrightarrow{\psi} K_0(\mathbb{Z}),$$

where ϕ is the map induced by $[C] \mapsto [Hoch_\bullet(C)]$, is non-trivial.

Proof. Take C to be a $\mathbb{Z}_2$ graded A_∞ algebra over $\mathbb{Z}$ such that $HH_0(C) = \mathbb{Z}$, $HH_1(C) = 0$, $HH_2(C) = \mathbb{Z}$, where $HH_\bullet(C)$ is the homology of $Hoch_\bullet(C)$. It is not hard to construct an example. As a geometric example take $C = FH_\bullet(L_0, L_0)$ where the latter is the Floer chain algebra of the equator L_0 in S^2, then the Hochschild homology $HH_\bullet(C)$ is isomorphic to the singular homology $H_\bullet(S^2, \mathbb{Z})$, [17].

So we get

$$\psi([Hoch_\bullet(C)]) = 2[\mathbb{Z}] \neq 0 \in K_0(\mathbb{Z}) \simeq \mathbb{Z}.$$

□

A further corollary of the proof above is:

Corollary 4.10. $[\widehat{\mathrm{Fuk}(S^2, \omega)}]$ *represents a non-trivial element in $K^{Cat,\mathbb{Z}_2}(\mathbb{Z})$.*

One natural question is how to produce interesting elements in $K_m^{Cat}(k)$ for higher m. We will show that there are somewhat exotic elements coming by way of symplectic geometry.

5. $|\mathcal{A}_k|$, $|\mathcal{A}_k^{\mathbb{Z}_n}|$ as representing spaces of Fukaya field theories.

We briefly review simplicial sets. We denote by Δ the simplex category:

- The set of objects of Δ is $\mathbb{N}$.
- $\hom_\Delta(n, m)$ is the set of non-decreasing maps $[n] \to [m]$, where $[n] = \{0, 1, \ldots, n\}$, with its natural order.

A simplicial set X is a functor

$$X : \Delta^{op} \to Set.$$

The set $X(n)$ is called the set of n-simplices of X. Δ^d will denote a particular simplicial set: the standard representable d-simplex, with

$$\Delta^d(n) = hom_\Delta(n, d).$$

Definition 5.1. For X a simplicial set, $\Delta(X)$ will denote the ***simplex category of*** X. This is the category s.t.:

- The set of objects $\operatorname{obj}\Delta(X)$ is the set of simplicial maps:
$$\Sigma : \Delta^d \to X, \quad d \geq 0.$$
- Morphisms $f : \Sigma_1 \to \Sigma_2$ are commutative diagrams in $s - Set$:

$$\begin{tikzcd} \Delta^d \arrow[r] \arrow[dr, "\Sigma_1"] & \Delta^n \arrow[d, "\Sigma_2"] \\ & X. \end{tikzcd} \tag{5.2}$$

The main construction of [15] has as input a Hamiltonian fibration $M \hookrightarrow P \to X$ [3], with (M, ω) a compact monotone symplectic manifold, e.g. $(\mathbb{CP}^n, \omega_{FS})$. And the output is a functor

$$F_P : \Delta(X_\bullet) \to A_\infty Cat_k^{\mathbb{Z}_2}, \tag{5.3}$$ [4]

well defined up to concordance of functors as defined below. We may understand F_P as giving a kind of topological field theory - the *Fukaya field theory* associated to P.

Definition 5.4. Let Y be a simplicial set and C any category. Let $F_i : \Delta(Y) \to C$, $i = 0, 1$ be functors. A ***concordance*** of F_0 to F_1 is a functor $\widetilde{F} : \Delta(Y \times I) \to C$ s.t. $\widetilde{F}|_{\Delta(Y \times \{0\})} = F_0$ and $\widetilde{F}|_{\Delta(Y \times \{1\})} = F_1$.

Then we obtain a functor:

$$(\widehat{F}_P = Perf \circ F_P) : \Delta(X_\bullet) \to A_\infty Cat^{\mathbb{Z}_2}, \tag{5.5}$$

again well defined up to concordance of functors, with image now consisting of pretriangulated idempotent complete A_∞ categories.

By construction $\widehat{F}_P$ takes all morphisms to quasi equivalences in $A_\infty Cat^{\mathbb{Z}_2}$. So we get an induced map:

$$\widehat{\mathcal{F}}_P : |\Delta(X_\bullet)| \to |wA_\infty Cat^{\mathbb{Z}_2}|, \tag{5.6}$$

where $|C|$ is shorthand for the geometric realization of the nerve of a category C.

The same construction applies to a concordance of functors, yielding a homotopy:

$$|\Delta(X_\bullet)| \times [0, 1] \to |wA_\infty Cat^{\mathbb{Z}_2}|. \tag{5.7}$$

Since $X \simeq |\Delta(X_\bullet)|$, we conclude that $\widehat{\mathcal{F}}_P$ induces a well defined class

$$[X \to |wA_\infty Cat^{\mathbb{Z}_2}|]$$

in $\pi_0(\mathrm{Maps}(X, |wA_\infty Cat^{\mathbb{Z}_2}|))$, where Maps is the space of continuous maps. By slight abuse we denote this class by $[\widehat{\mathcal{F}}_P]$.

To summarize, we obtain the following variation of the main theorem of [15].

Theorem 5.8. *Let $M \hookrightarrow P \to X$ be a Hamiltonian fibration with fiber (M, ω) a compact monotone symplectic manifold. Suppose that* $\mathrm{Fuk}(M, \omega)$ *satisfies the finiteness Condition* (3.2). *Then the assignment:*

$$[P] \mapsto [\widehat{\mathcal{F}}_P] \in \pi_0(\mathrm{Maps}(X, |w\mathcal{A}_k^{\mathbb{Z}_2}|)), \tag{5.9}$$

is well defined, where $[P]$ is the Hamiltonian isomorphism class of the fibration P.

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[3]Since the construction in [15] is carried out on the universal level, we do not need any smoothness assumptions on P here.

[4]In [15] we work with $k = \mathbb{Q}$ but this is only needed for inverting quasi equivalences. As we don't need to do this here, the construction works perfectly well over any commutative ring.

Proposition 5.10. $\mathrm{Fuk}(\mathbb{CP}^n, \omega_{FS})$ *is finite type, where the latter denotes the monotone Fukaya category,* [17].

Proof. Specifically, we take a concrete model of $\mathrm{Fuk}(\mathbb{CP}^n, \omega_{FS})$ with objects oriented Lagrangians Hamiltonian isotopic to the Clifford torus L_0. In this case it is automatic that $\mathrm{Fuk}(\mathbb{CP}^n, \omega_{FS})$ is quasi-isomorphic to the A_∞ algebra $HF(L_0, L_0)$, and finiteness follows. □

Example 3. More generally, given a compact Lie group G, G/T is always monotone with respect to some Kirillov-Kostant-Souriau symplectic structure. (The latter is well defined up deformation equivalence, see for instance [6]). It is natural to expect that G/T is finite type, the study of its monotone Lagrangians is carried out for example in [1].

Corollary 5.11. *There is a natural map:*

$$cl : B\,\mathrm{Ham}(\mathbb{CP}^n, \omega_{FS}) \to |w\mathcal{A}_k^{\mathbb{Z}_2}|, \tag{5.12}$$

well defined up to homotopy.

Proof. Let $\mathbb{CP}^n \hookrightarrow P \to B\,\mathrm{Ham}(\mathbb{CP}^n, \omega_{FS})$ be the tautological Hamiltonian fibration, that is the fiber bundle associated to the universal principal $\mathrm{Ham}(\mathbb{CP}^n, \omega_{FS})$ bundle over $B\,\mathrm{Ham}(\mathbb{CP}^n, \omega_{FS})$. By the Example 3 we may apply Theorem 5.8 to get a well defined up to homotopy map:

$$B\,\mathrm{Ham}(\mathbb{CP}^n, \omega_{FS}) \to |w\mathcal{A}_{\mathbb{Z}}^k|. \tag{5.13}$$

□

In [15] we deal with monotone symplectic manifolds, but all constructions can also be performed for compact Calabi-Yau manifolds. In this case, we can get $\mathbb{Z}$ grading, but we must work with a more specific type of coefficient ring. The standard type of choice is the universal Novikov field Λ, cf. Sheridan [16], whose elements are formal sums

$$\sum_{j=0}^{\infty} c_j r^{\lambda_j},$$

where $c_j \in \mathbb{C}$, and $\lambda_j \in \mathbb{R}$ is an increasing sequence of real numbers such that

$$\lim_{j\to\infty} \lambda_j = \infty.$$

In this case, we similarly obtain:

Theorem 5.14. *Let $M \hookrightarrow P \to X$ be a Hamiltonian fibration with fiber (M, ω) a compact Calabi-Yau symplectic manifold, s.t. $Fuk(M, \omega)$ satisfies the finiteness Condition 3.2. Then the assignment:*

$$[P] \mapsto [\widehat{\mathcal{F}}_P] \in \pi_0(\mathrm{Maps}(X, |w\mathcal{A}_\Lambda|)), \tag{5.15}$$

is well defined.

Definition 5.16. For brevity we call the symplectic manifolds in Theorem 3.5 respectively Theorem 3.5 *finite monotone type* and *finite Calabi-Yau type* respectively.

6. Hamiltonian elements in categorified algebraic K-theory

Now recall the definition of the infinite loop space (4.1). There is a natural map, see [26, Remark 8.3.2],

$$|w\mathcal{A}_k| \to \Omega |wS_\bullet(\mathcal{A}_k)| = K^{Cat}(k), \tag{6.1}$$

Similarly, there is a natural map

$$|w\mathcal{A}_k^{\mathbb{Z}_n}| \to K^{Cat,\mathbb{Z}_n}(k).$$

Combining with Corollary 5.11 we get:

Corollary 6.2. *For each commutative ring k, there is a natural homotopy class of a map:*

$$\mathrm{BHam}(\mathbb{CP}^n, \omega_{FS}) \to K^{Cat,\mathbb{Z}_2}(k),$$

and so homomorphisms:

$$\mathcal{U}_{m,*} : \pi_m(\mathrm{BHam}(\mathbb{CP}^n, \omega_{FS})) \to K_m^{Cat,\mathbb{Z}_2}(k).$$

As a first step towards the computation of the image of h_m we know that the map

$$\big(\mathbb{Z} \simeq \pi_4(BPU(2)) \simeq \pi_4(\mathrm{BHam}(S^2, \omega))\big) \to \pi_4(|w\mathcal{A}_{\mathbb{Z}}^{\mathbb{Z}_2}|, \mathrm{Fuk}\widehat{(S^2, \omega)}), \tag{6.3}$$

induced by the map cl, is injective, [13]. [5]

Thus, if we knew that

$$\pi_4(|w\mathcal{A}_{\mathbb{Z}}^{\mathbb{Z}_2}|, \mathrm{Fuk}\widehat{(S^2, \omega)}) \xrightarrow{\mathcal{U}_{4,*}} K_4^{Cat,\mathbb{Z}_2}(\mathbb{Z}) \tag{6.4}$$

was injective, we would obtain non-triviality of the image of $\mathcal{U}_{4,*}$. And we would have:

$$K_4^{Cat,\mathbb{Z}_2}(\mathbb{Z}) \text{ conditionally admits a } \mathbb{Z} \text{ injection.} \tag{6.5}$$

In general, there is no reason to expect $\mathcal{U}_{m,*}$ to be injective, but $m = 4$ is in the stable range of homotopy groups of $BPU(2)$. Here 'stable' means the range in which $BPU(n) \to BPU$ injects on homotopy groups. It is thus tempting to conjecture that there is a similar stabilization phenomenon for the more abstract mapping:

Conjecture 1.

$$\pi_{2k}(|w\mathcal{A}_{\mathbb{Z}}^{\mathbb{Z}_2}|, \mathrm{Fuk}(\widehat{\mathbb{CP}^{n-1}}, \omega_{FS}))) \xrightarrow{\mathcal{U}_{2k,*}} K_{2k}^{Cat,\mathbb{Z}_2}(\mathbb{Z}) \textit{ is injective, for } 2k \leq 2n.$$

The above is party motivated by [14, Theorem 1.2], which says that the topological stable range of $BPU(n)$ is related to the algebraic stable range of certain rational coefficients quantum characteristic classes, related to the Fukaya category theory underlying the construction of the Hamiltonian algebraic K-theory elements.

It should be emphasized that the latter conjecture is of algebraic topological nature, there is no analysis involved in its formulation (assuming $\mathrm{Fuk}(\mathbb{CP}^n, \omega_{st})$ is already given). On the other hand the content and the proof of the injectivity of (6.3) does involve geometric analysis.

[5] [13] does not work with pretriangulations, however pretriangulating does not change the relevant computation.

7. Hamiltonian elements in algebraic K-theory

It was our initial goal to obtain geometric elements of "classical" algebraic K-theory. First we have the "Calabi-Yau" case. There is a natural functor:

$$\phi : \mathcal{A}_k \to \mathrm{Perf}(k),$$

where $\mathrm{Perf}(k)$ is the Waldhausen category of perfect complexes over k, $\phi(C) = Hoch_\bullet(C)$ as in Section 4. Keller's theorem [5, Theorem 5.2] readily implies that this is a Waldhausen functor, that is it induces a map

$$\phi_* : K^{Cat}(k) \to K(k).$$

In particular, combining with Theorem 5.14 we get that an isomorphism class of a Hamiltonian fibration over S^k with compact Calabi-Yau type fiber determines an element of $K_k(\Lambda)$, as discussed in the introduction.

Proof of Theorem 1.2. We have a Waldhausen category $Ch_k^{\mathbb{Z}_2}$ of 2-periodic complexes of projective finite rank k-modules. It's Waldhausen K-theory infinite loop space will be denoted by $K^{\mathbb{Z}_2}(k)$. We again have the map:

$$\phi_* : K^{Cat,\mathbb{Z}_2}(k) \to K^{\mathbb{Z}_2}(k).$$

In particular, we obtain homotopy natural maps:

$$\mathrm{BHam}(\mathbb{CP}^n, \omega_{FS})) \to K^{Cat,\mathbb{Z}_2}(k) \xrightarrow{\phi_*} K^{\mathbb{Z}_2}(k), \quad \forall n \in \mathbb{N}.$$

Taking the composition with the natural map $BPU(n+1) \to \mathrm{BHam}(\mathbb{CP}^n, \omega)$, this proves the first part of the theorem.

If we knew that the monotone Fukaya category $\mathrm{Fuk}(G/T)$ is finite, we would likewise get homotopy natural maps:

$$\mathrm{BHam}(G/T) \to K^{Cat,\mathbb{Z}_2}(k) \xrightarrow{\phi_*} K^{\mathbb{Z}_2}(k).$$

This may not be known at the moment. However, we may directly construct the maps

$$\mathrm{BHam}(G/T) \to K^{\mathbb{Z}_2}(k),$$

as follows.

Let P denote denote the associated G/T bundle over $\mathrm{BHam}(G/T)$. The composition $Hoch_\bullet \circ F_P$ gives a functor $\Delta(X_\bullet) \to Ch_k^{\mathbb{Z}_2}$, taking all morphisms to quasi-equivalences, where $Hoch_\bullet$ denotes the Hochschild functor as previously. Again by the Waldhausen construction, there is an induced, natural up to homotopy map, $\mathrm{BHam}(G/T) \to K^{\mathbb{Z}_2}(k)$. And this specializes to give Theorem 1.2 in the introduction, using the natural map

$$BG \to \mathrm{BHam}(G/T).$$

□

8. Mirror Hamiltonian algebraic K-theory elements.

We may reformulate the famed homological mirror symmetry conjecture of Kontsevich in algebraic K-theory terms. In the case of Calabi-Yau manifolds, we propose that there is an equality of classes in $K_0^{Cat}(\Lambda)$:

$$[\mathrm{Fuk}(M)] = [D^b(M^{mirror})]$$

and

$$[\mathrm{Fuk}(M^{mirror})] = [D^b(M)],$$

where M, M^{mirror} are mirror dual, finite type Calabi-Yau manifolds. We have a construction of elements in $K_d^{Cat}(\Lambda)$ for all $d \geq 0$, which we may call "*A-model Hamiltonian algebraic K-theory elements*". That is the elements in $K_d^{Cat}(\Lambda)$, associated to Hamiltonian fibrations $M \hookrightarrow P \to S^d$, via Theorem 5.14.

We might then try to extend homological mirror symmetry to the context of higher degree Hamiltonian algebraic K-theory elements. Suppose we have a Hamiltonian G-action by a compact Lie group on a monotone symplectic manifold (M, ω). For example, the natural action of G on its co-adjoint orbit G/T, where T is the maximal torus. There is a proposal of Teleman [21], (see also his 2014 ICM address) on a extension of the homological mirror symmetry to such G gauged context. Denote by $G^\vee_{\mathbb{C}}$ the Langlands dual of the complexification $G_{\mathbb{C}}$. Then Teleman proposes that there is a B-model, (a type of Landau-Ginzburg model) with some "weak action" of $G^\vee_{\mathbb{C}}$. This has now been partially confirmed in some cases, [8].

In our setup, for each commutative ring k, we obtain a natural "A-model" homotopy class:

$$[f_M : BG \to K^{Cat, \mathbb{Z}_2}(k)], \tag{8.1}$$

as the composition $BG \to \mathrm{BHam}(M, \omega) \to K^{Cat}(\Lambda)$. To mimic Teleman's idea, we may propose that there is also a natural "B-model" homotopy class

$$[f_{(G \curvearrowright M)^{mirror}} : B(G^\vee_{\mathbb{C}}) \to K^{Cat, \mathbb{Z}_2}(k)], \tag{8.2}$$

assuming $G \curvearrowright M$ has a suitable Teleman mirror $(G \curvearrowright M)^{mirror}$. We may then conjecture that the induced maps:

$$\pi_\bullet(BG) \to K^{Cat, \mathbb{Z}_2}_\bullet(k),$$
$$\pi_\bullet(B(G^\vee_{\mathbb{C}})) \to K^{Cat, \mathbb{Z}_2}_\bullet(k)$$

have the same image. What is mysterious at the moment is not the possible correspondence itself but how (8.2) might be constructed.

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