QUANTUM MASLOV CLASSES

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Abstract. We give a construction of "quantum Maslov characteristic classes", generalizing to higher dimensional cycles and furthermore higher homotopy groups the Hu-Lalonde-Seidel morphism. We also state a conjecture extending this to an A_∞ functor from the exact path category of the space of monotone Lagrangian branes to the Fukaya category. Quantum Maslov classes are used here for the study of Hofer geometry of Lagrangian equators in S^2, giving a rigidity phenomenon for the Hofer metric 2-systole, which stands in contrast to the flexibility phenomenon of the closely related Hofer metric girth studied by Rauch [13], in the same context of Lagrangian equators of S^2. More applications appear in [14].

1. Introduction

There are numerous results in symplectic geometry which express a kind of symplectic flexibility vs rigidity phenomena. But one novelty here is that this rigidity/flexibility will involve Hofer geometry of Lagrangian submanifolds. In fact, we will see this just for the simplest example of Lagrangian equators of S^2.

Our main method is a higher dimensional generalization of the Hu-Lalonde-Seidel morphism [7], that we call Quantum Maslov classes.

Let $Lag(M)$ denote the space of monotone Lagrangian branes: in particular closed, oriented, spin, monotone Lagrangian submanifolds of a symplectic manifold (M, ω). The Hu-Lalonde-Seidel morphism in one basic form is a homomorphism:

$$\Psi : \pi_1(Lag(M), L_0) \to FH^\times(L_0, L_0), \tag{1.1}$$

where $FH^\times(L_0, L_0)$ denotes the group of invertible elements in the $\mathbb{Z}_2$ graded Floer homology algebra.

Denote by $\mathcal{P}(L_0, L_1)$ the space of exact paths in $Lag(M)$ from L_0 to L_1, with its natural C^∞ topology, see Section 4.1.1. We set $\Omega_{L_0} Lag(M) := \mathcal{P}(L_0, L_0)$.

Denote by $Ob_d(X)$ the oriented bordism groups of a space X. The following is proved in Section 3.

Theorem 1.2. *For all d, there is a natural and (generally) non-trivial group homomorphism:*

$$\Psi : Ob_d(\mathcal{P}(L_0, L_1)) \to FH(L_0, L_1).$$

Furthermore, the restriction of Ψ *to* $\pi_d(\mathcal{P}(L_0, L_1), p_0)$ *determines a group homomorphism:*

$$\Psi : \pi_d(\mathcal{P}(L_0, L_1), p_0) \to FH(L_0, L_1) \ \forall d \geq 1,$$

where p_0 *is any base point.*

The extension furthermore, determines a certain functor to the Donaldson-Fukaya category. But we will not give details of this here, as it is not needed for the main geometric application. See Conjecture 2 for a formal statement, on the level of an A_∞ functor to the Fukaya category.

Let $Eq(S^2) \subset Lag(S^2)$ be the subspace of great circles. Note that $Eq(S^2)$ is naturally diffeomorphic to S^2. The main thrust of the paper is the following result implied by Theorem 4.13.

Theorem 1.3. *For $M = S^2$ and L_0 the standard equator, the composition map:*

$$\mathbb{Z} \simeq \pi_2(\Omega_{L_0} Eq(S^2)) \to \pi_2(\Omega_{L_0} Lag(M)) \xrightarrow{\Psi} FH(L_0, L_0),$$

is injective.

The result is crucial in [14]. The proof requires some Riemannian and Hofer geometry and infinite dimensional Morse theory.

We will use non-triviality of Ψ for our Hofer geometric application. This story is formally similar to author's use of quantum characteristic classes in Hofer geometry, [15]. However, unlike the case of [15], when we try to apply the calculation of Ψ to the study of Hofer geometry of Lagrangians, an unexpected new complexity arises. To get anything interesting (for $d > 0$) we need certain tautness conditions on families of Lagrangians. This gives rise to new geometric structures called taut Hamiltonian structures, whose theory we partly develop here.

1.1. **Taut conditions and Hofer geometry.** The basic taut condition is the following.

Definition 1.4. *Two smooth loops*

$$p_0, p_1 : S^1 \to Lag(M)$$

are said to be **taut concordant** *if the following holds:*

- *There is a smooth fiber-wise Lagrangian sub-fibration*
$$\mathcal{L} \subset Cyl \times M, \quad Cyl = S^1 \times [0, 1],$$
of the trivial fibration $Cyl \times M \to Cyl$, satisfying $\mathcal{L}_{(\theta,0)} = p_0(\theta)$ and $\mathcal{L}_{(\theta,1)} = p_1(\theta)$ for all $\theta \in S^1$.
- *There is a Hamiltonian connection $\mathcal{A}$ on the fibration $Cyl \times M$, preserving $\mathcal{L}$, such that the coupling form $\Omega_{\mathcal{A}}$ of $\mathcal{A}$ vanishes on $\mathcal{L}$. See Section 2.1 for the definition of coupling forms.*

It is easy to see that taut concordance is an equivalence relation (take connections constant near boundary and glue).

Example 1. Lemma 4.5 implies that any loop in $Eq(S^2)$ is taut concordant to a constant loop.

Example 2. Let us call a smooth loop $o : S^1 \to Lag(M)$ ***strictly exact*** if there is a smooth loop $\widetilde{o} : S^1 \to \mathrm{Ham}(M, \omega)$, s.t. $o(t) = \widetilde{o}(t)(o(0))$, where the right hand side means apply a diffeomorphism to get a new Lagrangian submanifold. Then any strictly exact o, with $\widetilde{o}$ contractible is taut homotopic to a constant loop. To see this, let $H : S^1 \times [0, 1] \to \mathrm{Ham}(M, \omega)$ be a null-homotopy of $\widetilde{o}$ to the constant loop at the identity. It is then elementary to construct a flat Hamiltonian connection $\mathcal{A}$ on $Cyl \times M$, whose holonomy path over $S^1 \times \{0\}$ is the constant loop at the id and the holonomy path over $S^1 \times \{1\}$ given by $\widetilde{o}$. If we fix any Lagrangian L_0 in the fiber of $Cyl \times M \to Cyl$ over $(0, 0)$, then by the flatness condition of $\mathcal{A}$, this uniquely extends to an $\mathcal{A}$-invariant Lagrangian sub-fibration $\mathcal{L}_0$ of $Cyl \times M$. This satisfies: $\mathcal{L}_0$ over $S^1 \times \{1\}$ is o and $\mathcal{L}_0$ over $S^1 \times \{0\}$ is the constant loop at L_0.

1.1.1. *Lagrangian equators in S^2.* Let $L_0 \subset S^2$ now denote the standard equator.

The natural embedding

$$i : \Omega_{L_0} Eq(S^2) \hookrightarrow \Omega_{L_0} Lag(S^2),$$

has image in $\Omega^{taut}_{L_0} Lag(S^2)$ by the Example 1.

Set $b = i_*(gen)$ for

$$gen \in \pi_2(\Omega_{L_0} Eq(S^2), p_{L_0}) \simeq \pi_3(S^2) \simeq \mathbb{Z},$$

the generator, and where p_{L_0} denotes the constant loop at L_0.

Theorem 1.5. *We have the following identity for the 2-systole with respect to L^+:*

$$\min_{f\in b}\max_{s\in S^2} L^+(f(s)) = 1/2\cdot \mathrm{Vol}(S^2,\omega),$$

where L^+ denotes the Lagrangian positive Hofer length functional, as defined in Section 4.1. And where f has image in the space of loops taut concordant to the constant loop. Furthermore,

$$(\mathbb{Z}\simeq \pi_3(Eq(S^2)))\to \pi_3(Lag(S^2)) \text{ is an injection}[1].$$

The following corollary is still non-trivial and is not implied by other methods as far as I know.

Corollary 1.6.

$$\min_{f\in b}\max_{s\in S^2} L^+(f(s)) = 1/2\cdot \mathrm{Vol}(S^2,\omega),$$

where f has image in the subspace of strictly exact loops.

Proof. This readily follows by Example 2, and the theorem above. □

For contrast, suppose we measure a related quantity of the "girth" as in Rauch [13]. That is, let

$$inc : Eq(S^2)\to Lag(S^2)$$

be the natural inclusion. Define

$$girth_{L^+}(inc_* a)$$

to be the infimum of L^+-diameter of a representative of $inc_* a\in \pi_2(Lag(S^2),L_0)$, for $a\in \pi_2(Eq(S^2))\simeq \mathbb{Z}$ the generator. Rauch shows that

$$girth_{L^+}(inc_* a) < girth_{inc^* L^+}(a),$$

where the right-hand side is defined analogously using the functional $inc^* L^+ = L^+\circ inc$. Indeed, it may be that $girth_{L^+}(inc_* a) = 0$. I think the latter is unlikely, it would in particular disprove the Lagrangian version of the injectivity radius conjecture of [9].

To summarize, passing from classical equators to Lagrangian equators, we see a squeezing phenomenon for girth. On the other hand, our theorem says that this kind of squeezing cannot happen at all for the 2-systole, provided we work with taut families. So we get a flexibility vs rigidity phenomenon in Hofer geometry.

The construction of Rauch should also show the flexibility of the 2-systole as soon as we remove the tautness assumption. The following is a minor reformulation of the main result in [13].

Conjecture 1.

$$\inf_{f\in b}\{\max_{s\in S^2} L^+(f(s))\} < 1/2\cdot \mathrm{Vol}(S^2,\omega). \tag{1.7}$$

Corollary 1.8. *Assume the conjecture above, then there are loops $S^1\to Lag(S^2)$ which are homotopic to [2], but not taut concordant to the constant loop at L_0.*

Proof. Let $f : S^2\to \Omega_{L_0} Lag(S^2)$, represent b, and satisfy

$$\max_{s\in S^2} L^+(f(s)) < 1/2\cdot \mathrm{Vol}(S^2,\omega).$$

By Theorem 1.5 the image of f cannot be in subspace of loops taut concordant to the constant loop. Since every $f(s)$ is homotopic to the p_{L_0}, by the condition that f is based at p_{L_0}, the conclusion immediately follows. □

[1] This appears to be unknown. It should be possible to deduce this via Smale's theorem on homotopy type of $\mathrm{Diff}(S^2)$, but we don't use this.

[2] In fact $\pi_1(Lag(S^2)) = 0$, but this is not elementary.

Another question:

Question 1. It is shown in [17] that a certain semi-classical limit of quantum characteristic classes are the Chern classes. Is there some semi-classical limit for the quantum Maslov classes?

1.2. **Acknowledgements.** I am grateful to Itamar Rauch and Misha Khanevsky for sharing a manuscript which inspired writing up this note.

2. Hamiltonian fibrations and taut structures

We collect here some preliminaries on moduli spaces of holomorphic sections of fibrations with Lagrangian boundary constraints, and the closely related curvature bounds. Some of the generality here is a slightly excessive for the main applications, but it is used in [14], and it does not substantially increase volume.

2.1. **Coupling forms.** We refer the reader to [10, Chapter 6] for more details on what follows. We will suppose throughout that (M, ω) is a closed symplectic manifold, and later on monotone. A Hamiltonian fibration is a smooth fiber bundle

$$M \hookrightarrow P \to X,$$

with structure group $\mathcal{H} = \mathrm{Ham}(M, \omega)$ with its C^∞ Frechet topology. A ***Hamiltonian connection*** is just an Ehresmann $\mathcal{H}$ connection on this fiber bundle. $\mathcal{H}$ trivializations will be called Hamiltonian bundle trivializations. $\mathcal{H}$ bundle maps will be called Hamiltonian fibration maps or Hamiltonian bundle maps.

A *coupling form*, as defined in [5], for a Hamiltonian fibration $M \hookrightarrow P \xrightarrow{p} X$, is a closed 2-form Ω on P whose restriction to fibers coincides with ω and that has the property:

$$\int_M \Omega^{n+1} = 0,$$

with integration being integration over the fiber operation.

Such a 2-form determines a Hamiltonian connection, by declaring horizontal spaces to be Ω-orthogonal spaces to the vertical tangent spaces. A coupling form generating a given connection $\mathcal{A}$ is unique. A Hamiltonian connection $\mathcal{A}$ in turn determines a coupling form $\Omega_{\mathcal{A}}$ as follows. First we ask that $\Omega_{\mathcal{A}}$ induces the connection $\mathcal{A}$ as above. This determines $\Omega_{\mathcal{A}}$ up to values on $\mathcal{A}$-horizontal lifts $\widetilde{v}, \widetilde{w} \in T_p P$ of $v, w \in T_x X$. We specify these values by the formula

$$\Omega_{\mathcal{A}}(\widetilde{v}, \widetilde{w}) = R_{\mathcal{A}}(v, w)(p), \tag{2.1}$$

where $R_{\mathcal{A}}$ is the lie algebra valued curvature 2-form of $\mathcal{A}$. Specifically, for each x, $R_{\mathcal{A}}|_x$ is a 2-form valued in $C^\infty_{norm}(p^{-1}(x))$ - the space of 0-mean normalized smooth functions on $p^{-1}(x)$.

2.2. **Hamiltonian structures on fibrations.** Let S be a Riemann surface with boundary, with some finite number of punctures on the boundary. As part of the data, we ask for a collection of *strip end charts.* This is the data of holomorphic diffeomorphisms

$$[0, 1] \times (0, \infty) \to S,$$

at so called positive ends. And holomorphic diffeomorphisms

$$[0, 1] \times (-\infty, 0) \to S,$$

at so called negative ends. This mirrors the discussion in [19, Section 3.2].

Let $M \hookrightarrow \widetilde{S} \xrightarrow{pr} S$ be a Hamiltonian fibration, with model fiber a monotone symplectic manifold (M, ω), with distinguished Hamiltonian bundle trivializations

$$[0, 1] \times (0, \infty) \times M \to \widetilde{S}$$

at the positive ends, and with distinguished Hamiltonian bundle trivializations

$$[0,1] \times (-\infty, 0) \times M \to \widetilde{S},$$

at the negative ends, which are the over strip end charts as above. These are collectively called called ***end bundle charts***. Given the structure of such Hamiltonian bundle trivializations we say that $\widetilde{S}$ has ***end structure***.

Definition 2.2. *Let*

$$\mathcal{L} \subset (\widetilde{S}|_{\partial S} = pr^{-1}(\partial S)) \to \partial S$$

be a Lagrangian sub-bundle, with model fiber a Lagrangian brain, or more specifically an object as in [19, Section 5], *(in particular a closed, spin, oriented, monotone Lagrangian submanifold). We say that* $\mathcal{L}$ **respects the end structure** *if* $\mathcal{L}$ *is a constant sub-bundle in the end bundle charts above.*

For future use:

Notation 1. Let $\mathcal{L}$ respect the end structure of $\widetilde{S}$ as above. At the i'th end, in the end bundle chart as above, let L_i^j denote the fibers (which are by assumption t independent) of $\mathcal{L}$ over

$$\{j\} \times \{t\},\ j = 0, 1.$$

Let us write $(\widetilde{S}, \mathcal{L})$ for a pair as above.

Definition 2.3. *We denote by* $\mathcal{G}(\widetilde{S}, \mathcal{L})$ *the group of Hamiltonian* M*-bundles maps of* $\widetilde{S}$ *that preserve* $\mathcal{L}$ *and that are trivial in the end bundle charts.*

Definition 2.4. *Let* $(\widetilde{S}, \mathcal{L})$ *be as above with ends* e_i *indexed by* I*. For a choice of Hamiltonian connections* $\mathcal{A}_i$ *on* $[0,1] \times M$*,* $i \in I$*, we say that a Hamiltonian connection* $\mathcal{A}$ *on* $\widetilde{S}$ *is* **compatible** *with* $\mathcal{L}$*,* $\{\mathcal{A}_i\}$ *if the following holds.*

1. *In the end bundle chart at the* e_i *end,* $\mathcal{A}$ *is flat and* $\mathbb{R}$*-translation invariant. Moreover, in this end bundle chart it has the form* $\overline{\mathcal{A}}_i$*, where* $\overline{\mathcal{A}}_i$ *denotes the* $\mathbb{R}$*-translation invariant extension of the Hamiltonian connection* $\mathcal{A}_i$ *to* $(0,\infty) \times \mathbb{R} \times M$*, in the case of a positive end, similarly for negative.*
2. *It is* $\mathcal{L}$-**exact***, which means that* $\mathcal{A}$ *preserves* $\mathcal{L}$ *(this entails that the horizontal spaces of* $\mathcal{A}$ *are tangent to* $\mathcal{L}$*).*

For $\mathcal{A}$ compatible with $\{\mathcal{A}_i\}$ as above, a family $\{j_z\}$ of fiber wise ω-compatible almost complex structures on $\widetilde{S}$ will be said to ***respect the end structure*** if the following holds. At each end, in the end bundle chart, the family $\{j_z\}$ is $\mathbb{R}$-translation invariant and is admissible with respect to $\mathcal{A}_i$, in the sense of [Definition 5.3][19]. The data $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A}, \{j_z\})$, as above, with $\mathcal{A}$ compatible and $\{j_z\}$, respecting the end structure, will be called a ***Hamiltonian structure***.

We will normally suppress $\{j_z\}$ in the notation and elsewhere for simplicity, as it will be purely in the background in what follows, (we do not need to manipulate this family explicitly).

2.3. Relative section classes of Hamiltonian structures.

Definition 2.5. *Let* $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ *be a Hamiltonian structure, we say that a smooth section* σ *of* $\widetilde{S} \to S$ *is* **asymptotically flat** *if the following holds. At each end* e_i *of* S*,* σ C^1*-converges to an* $\mathcal{A}$*-flat section. Specifically, in the end bundle chart at a positive end, this means that there is a* $\mathcal{A}$*-flat section*

$$\widetilde{\sigma} : [0,1] \times (0,\infty) \to [0,1] \times (0,\infty) \times M,$$

so that for every $\epsilon > 0$ there is a $t > 0$ so that

$$d_{C^1}(\widetilde{\sigma}, \sigma|_{[0,1]\times[t,\infty)}) < \epsilon.$$

(Similarly for a negative end.)

Note that the above definition implies that

$$\lim_{s\mapsto\infty} \sigma|_{[0,1]\times\{s\}} = \gamma^i,$$

for some $\mathcal{A}_i$-flat sections γ_i of $[0,1]\times M$, where the limit is the C^1 limit. (Similarly for negative ends.) So we can say that σ is ***asymptotic*** to γ^i at the e_i end, and that γ^i is the ***asymptotic constraint*** of σ at the e_i end.

Definition 2.6. *Given a pair of asymptotically flat sections σ_1, σ_2, with boundary in $\mathcal{L}$, we say that they have the same* **relative class** *if:*

- *They are asymptotic to the same flat sections at each end. (In the sense above.)*
- *They are homologous relative to the boundary conditions and relative to the asymptotic constraints at the ends.*

The set of relative classes will be denoted by $H_2^{sec}(\widetilde{S}, \mathcal{L})$ or by $H_2^{sec}(\Theta)$.

2.4. **Families of Hamiltonian structures.**

Definition 2.7. *A* **family Hamiltonian structure***, consists of the following:*

1. *A smooth, connected, compact, oriented manifold $\mathcal{K}$, possibly with boundary or corners.*
2. *A pair $(\widetilde{S}, \mathcal{L})$ as in the paragraph after Definition 2.2, and an associated bundle $P \times_{\mathcal{G}(\widetilde{S},\mathcal{L})} \widetilde{S}$ for some principal $\mathcal{G}(\widetilde{S}, \mathcal{L})$ bundle P over $\mathcal{K}$. We will write*
$$\widetilde{S} \hookrightarrow \widetilde{\boldsymbol{S}} \xrightarrow{p_1} \mathcal{K}$$
for this associated bundle.
3. *A choice of connections $\{\mathcal{A}_i\}$ on $[0,1]\times M$ corresponding to each end e_i of S.*
4. *A complex structure j_r, $r \in \mathcal{K}$, on the base S of the fibration $(\widetilde{S}_r = p_1^{-1}(r)) \to S$, so that this is smooth in r in the standard sense.*
5. *A smooth section $\{\mathcal{A}_r\}$ of the associated bundle over $\mathcal{K}$, whose fiber is the space of Hamiltonian connections on $\widetilde{S}$ compatible with $\mathcal{L}$, $\{\mathcal{A}_i\}$. Note that the latter fiber is contractible, being affine and non-empty.*

The above entails that for each $r \in \mathcal{K}$ we have a Hamiltonian structure $(\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)$, such that this fits into a smooth family. We will write $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ for this data, $\mathcal{K}$ may be omitted from notation when it is implicit. The connections $\{\mathcal{A}_i\}$ will also be implicit, as they are determined by $\mathcal{A}_r$.

Note that there is also a natural Hamiltonian fibration:

$$M \hookrightarrow \widetilde{\boldsymbol{S}} \to \mathcal{K} \times S,$$

induced by the data above. This has a fiber-wise Lagrangian subfibration over $\mathcal{K} \times \partial S$, which is is denoted by

$$L \hookrightarrow \boldsymbol{L} \to \mathcal{K} \times \partial S.$$

Terminology 1. We will usually just say Hamiltonian structure instead of family Hamiltonian structure. The distinction between the two is clear from context and notation.

Definition 2.8. *Let $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}$ be a Hamiltonian structure. In the notation above, if in addition there exists a Hamiltonian connection $\mathcal{A}$ on $M \hookrightarrow \widetilde{\boldsymbol{S}} \to \boldsymbol{S}$, extending $\{\mathcal{A}_r\}$, and so that $\Omega_{\mathcal{A}}$ vanishes on $\boldsymbol{L}$, we will say that $\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}$ is a* **hyper taut Hamiltonian structure***.*

2.5. **Moduli spaces of sections of Hamiltonian structures.** Let $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be a Hamiltonian structure. For a section σ of $\widetilde{S}$ define its vertical L^2 energy or ***Floer energy*** by

$$e(\sigma) := \int_S |\pi_{vert} \circ d\sigma|^2,$$

$$\pi_{vert} : T\widetilde{S} \to T^{vert}\widetilde{S}$$

is the $\mathcal{A}$-projection, for $T^{vert}\widetilde{S}$ the vertical tangent bundle of $\widetilde{S}$, that is the kernel of the projection $T\widetilde{S} \to TS$.

As in [19], we define an almost complex structure $J(\mathcal{A})$ on $\widetilde{S}$ determined by $\mathcal{A}$, $\{j_z\}$, and the complex structure j on S as follows.

- $J(\mathcal{A})$ preserves the $\mathcal{A}$-horizontal distribution of $\widetilde{S}$.
- The projection map $\widetilde{S} \to S$ is $J(A)$-holomorphic.
- The restriction of $J(\mathcal{A})$ to each fiber M_z of $\widetilde{S}$ over $z \in S$ is j_z.

We say that $J(\mathcal{A})$ is ***induced*** by $\mathcal{A}$, $\{j_z\}$.

By standard Floer theory any finite Floer energy $J_{\mathcal{A}}$-holomorphic section with boundary on $\mathcal{L}$ is an asymptotically flat section, and hence has an associated class in $H_2^{sec}(\widetilde{S}, \mathcal{L})$. Note that for any $J(\mathcal{A})$-holomorphic σ we have an identity:

$$e(\sigma) = \int_S \sigma^* \Omega_{\mathcal{A}}.$$

We leave to the reader to verify that $\Omega_{\mathcal{A}}$ vanishes on $\mathcal{L}$, using the conditions that $\mathcal{A}$ preserves $\mathcal{L}$, and that $\mathcal{L}$ is a fiber-wise Lagrangian distribution. So the standard energy controls apply, once we establish Lemma 2.21 further ahead. We then deduce the standard Gromov-Floer compactification structure on spaces of $J(\mathcal{A})$-holomorphic sections in a fixed class $A \in H_2^{sec}(\widetilde{S}, \mathcal{L})$.

We then set $\overline{\mathcal{M}}(\Theta, A)$ to be the Gromov-Floer compactification of the space of finite Floer energy, class $A \in H_2^{sec}(\widetilde{S}, \mathcal{L})$ $J(\mathcal{A})$-holomorphic sections σ of $\widetilde{\mathcal{S}}$, with boundary on $\mathcal{L}$.

2.5.1. *Family version.* More generally, let $\{\Theta_r\} = \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ be a family Hamiltonian structure. Pick an Ehresmann, $\mathcal{G}(\widetilde{S}, L)$ structure fiber bundle connection $\mathcal{B}$ on $\widetilde{S} \hookrightarrow \widetilde{\boldsymbol{S}} \xrightarrow{p_1} \mathcal{K}$. Define $Sec(\{\Theta_r\})$ to be the set of equivalence classes of pairs (A, r) for $A \in H_2^{sec}(\Theta_r)$, where $(A, r) \sim (B, r')$ if B is the $\mathcal{B}$ parallel transport of A over some path from r to r'. We may by abuse of notation just write A for an element of $Sec(\{\Theta_r\})$.

Let

$$\overline{\mathcal{M}}(\{\Theta_r\}, A)$$

be the Gromov-Floer compactification of the space of pairs (σ, r), $r \in \mathcal{K}$ with σ a $J(\mathcal{A}_r)$-holomorphic, finite Floer energy, class $A \in Sec(\{\Theta\})$ section of $\widetilde{\mathcal{S}}_r$, with boundary on $\mathcal{L}_r$. Here $J(\mathcal{A}_r)$ is as in Section above, defined with respect to the complex structure j_r on S_r. The class of (σ, r) is the class of $([\sigma], r)$ in $Sec(\{\Theta_r\})$.

2.5.2. *Regularity.* Let $\{\Theta_r = (\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r)\}$ be a Hamiltonian structure, then for each end e_i of S_r we have a Floer chain complex

$$CF(\mathcal{A}_i) := CF(L^0_i, L^1_i, \mathcal{A}_i, \{j_z\}),$$

generated over $\mathbb{Q}$ by $\mathcal{A}_i$-flat sections of $[0,1] \times M$, with boundary on L^0_i, L^1_i. This chain complex is defined as in [19, Section 6.1].

Definition 2.9. *We say that $\{\Theta_r\}$ is A-**regular** if:*

- *The pairs $(\mathcal{A}_i, \{j_z\})$ are regular so that the Floer chain complexes $CF(\mathcal{A}_i)$ are defined.*
- *$\mathcal{M}(\{\Theta_r\}, A)$ is regular, (transversely cut out).*

*And we say that $\{\Theta_r\}$ is **regular** if it is A-regular for all A. We say that $\{\Theta_r\}$ is A-**admissible** if there are no elements*

$$(\sigma, r) \in \overline{\mathcal{M}}(\{\Theta_r\}, A),$$

for r in a neighborhood of the boundary of $\mathcal{K}$.

Definition 2.10. *Suppose that $\mathcal{K}_i$, $i = 1, 2$ have no boundary. Given a pair $\{\Theta^i_r\} = \{\widetilde{S}^0_r, S^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}_{\mathcal{K}_i}$, of Hamiltonian structures we say that they are **bordant** if the following holds. There is a Hamiltonian structure*

$$\mathcal{T} = \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{B}},$$

with an oriented diffeomorphism (in the natural sense, preserving all structure)

$$\{\widetilde{S}^0_r, S^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}_{\mathcal{K}^{op}_0} \sqcup \{\widetilde{S}^1_r, S^1_r, \mathcal{L}^1_r, \mathcal{A}^1_r\}_{\mathcal{K}_1} \to \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\partial \mathcal{B}},$$

*where op denotes the opposite orientation. If $\mathcal{K}_0 = \mathcal{K}_1$, and possibly with boundary we say that the structures above are **concordant** if there is a $\mathcal{T}$ as above with $\mathcal{B} = \mathcal{K} \times [0,1]$. And there is an oriented diffeomorphism (again in the natural sense)*

$$\{\widetilde{S}^0_r, S^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}_{\mathcal{K}^{op}_0} \sqcup \{\widetilde{S}^1_r, S^1_r, \mathcal{L}^1_r, \mathcal{A}^1_r\}_{\mathcal{K}_1} \to \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times \partial [0,1]}.$$

Definition 2.11. *We say that a Hamiltonian structure $\{\Theta_r\} = \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$ is **taut** if the following holds. For any pair $r_1, r_2 \in \mathcal{K}$, Θ_{r_1} is concordant to Θ_{r_2} by a concordance $\{\widetilde{T}_\tau, T_\tau, \mathcal{L}'_\tau, \mathcal{A}'_\tau\}_{[0,1]}$, which is a hyper taut Hamiltonian structure.*

Definition 2.12. *Given an A-admissible pair $\{\Theta^i_r\}$, $i = 1, 2$, of Hamiltonian structures, we say that they are A-**admissibly concordant** if there is a Hamiltonian structure*

$$\{\mathcal{T}_r\} = \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K} \times [0,1]},$$

*which furnishes a concordance, and s.t. there are no elements $(\sigma, r) \in \overline{\mathcal{M}}(\{\Theta_r\}, A)$, for $r \in \partial \mathcal{K} \times [0,1]$. If this concordance is in addition a taut Hamiltonian structure, then we say that these pairs are **A-admissibly taut concordant**.*

Lemma 2.13. *Let $\{\Theta_r\} = \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}$ be A-regular and A-admissible, with S_r having one distinguished negative end e_0, and no positive ends, and let γ_0 be the asymptotic constraint of A at the e_0 end. Define*

$$ev(\{\Theta_r\}, A) = \#\mathcal{M}(\{\Theta_r\}, A) \cdot \gamma_0 \in CF(\mathcal{A}_0),$$

where $\#\mathcal{M}(\{\Theta_r\}, A)$ means signed count of elements when the dimension is 0, and is otherwise set to be zero. Furthermore, suppose that $CF(\mathcal{A}_0)$ is perfect. Then the count $\#\mathcal{M}(\{\Theta_r\}, A)$ depends only on the A-admissible concordance class of $\{\Theta_r\}$, that is the homology class of ev_A is an invariant of the A-admissible concordance class of $\{\Theta_r\}$.

Proof. Suppose we are given an A-admissible concordance (which we may assume to be regular)

$$\mathcal{T} = \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K}\times[0,1]},$$

between Hamiltonian structures $\{\Theta^0_r\}$ and $\{\Theta^1_r\}$. Then we get a one dimensional compact moduli space $\mathcal{M}(\mathcal{T}, A)$. By assumption on the perfection of $CF(\mathcal{A}_0)$, boundary contributions from Floer degenerations cancel out, so that the boundary is:

$$\partial\mathcal{M}(\mathcal{T}, A) = \mathcal{M}(\{\Theta^0_r\}^{op}, A) \sqcup \mathcal{M}(\{\Theta^1_r\}, A)$$

where *op* denotes opposite orientation. From which the result follows. □

Definition 2.14. *Let $\{\Theta_r\} = \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}$, be a regular Hamiltonian structure, with $\mathcal{K}$ possibly with boundary, and with S_r having one distinguished negative end e_0 and no positive ends. Define:*

$$ev(\{\Theta_r\}) = \sum_A ev(\{\Theta_r\}, A) \in CF(\mathcal{A}_0), \tag{2.15}$$

where the sum is over all classes $A \in Sec(\{\Theta_r\})$.

In general $ev(\{\Theta_r\})$ is not closed, but we have:

Lemma 2.16. *Let $\{\Theta_r\}_{r\in\mathcal{K}}$ be regular, with $\mathcal{K}$ having no boundary, with S_r having one distinguished negative end e_0 and no positive ends. Then $ev(\{\Theta_r\})$ is a cycle, whose homology class depends only on the bordism class of $\{\Theta_r\}$.*

Proof. We only sketch the proof since this formally analogous to the standard proof that Floer continuation maps are chain homotopy maps.

Set:

$$\mathcal{M}^1 = \cup_A \mathcal{M}(\{\Theta_r\}, A),$$

where the sum is over all A such that the expected dimension of $\mathcal{M}(\{\Theta_r\}, A)$ is one. As usual this is a finite sum by monotonicity. Then $\mathcal{M}^1$ is a one dimensional oriented manifold with boundary.

Then

$$\#\partial\mathcal{M}^1 = 0,$$

where this is the signed count of elements of an oriented 0-dimensional manifold. As $\mathcal{K}$ has no boundary, this is a signed count of holomorphic buildings (broken flow lines) with a pair of components. One component is an element σ of $\mathcal{M}(\{\Theta_r\}, A')$, for some A', s.t. the expected dimension of $\mathcal{M}(\{\Theta_r\}, A')$ is zero, i.e. contributing to $ev(\{\Theta_r\})$. The other component corresponds to a contribution to the Floer boundary of $\gamma_{A'}$, where the latter is the asymptotic constraint of A'.

As usual all such boundary contributions to $\partial\mathcal{M}^1$ that can happen do happen (by a standard gluing argument). It then readily follows, since $\#\partial\mathcal{M}^1 = 0$, that the Floer differential of $ev(\{\Theta_r\})$ is zero.

The second part of the lemma is analogous. Let

$$\mathcal{T} = \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{B}},$$

be a bordism (which we may assume to be regular) between Hamiltonian structures $\{\Theta^0_r\}$ and $\{\Theta^1_r\}$, with the latter as in the hypothesis of the lemma.

Set

$$\mathcal{M}^1 = \cup_A \mathcal{M}(\mathcal{T}, A),$$

Then again this is a one dimensional oriented manifold with boundary. Analyzing $\partial\mathcal{M}^1$, and since $\#\partial\mathcal{M}^1 = 0$, we get that

$$\partial\, ev((\{\mathcal{T}\})) = ev(\{\Theta^1_r\}) - ev(\{\Theta^0_r\}).$$

And this finishes the proof.

□

2.6. **Area of fibrations.**

Definition 2.17. *For a Hamiltonian connection $\mathcal{A}$ on a bundle $M \hookrightarrow \widetilde{S} \to S$, with S a Riemann surface, define a 2-form $\alpha_{\mathcal{A}}$ on S by:*

$$\alpha_{\mathcal{A}}(v, jv) := |R_{\mathcal{A}}(v, jv)|_+, \tag{2.18}$$

where $v \in T_z S$, $R_{\mathcal{A}}(v, w)$ as before identified with a zero mean smooth function on the fiber $\widetilde{S}_z$ over z and where $|\cdot|_+$ is operator: $|H|_+ = \max_{\widetilde{S}_z} H$, i.e. the "positive Hofer norm".

And define

$$\text{energy}(\mathcal{A}) := \int_S \alpha_{\mathcal{A}}. \tag{2.19}$$

Note that if $\Omega_{\mathcal{A}}$ is the coupling form of $\mathcal{A}$, as before, then $\Omega_{\mathcal{A}} + \pi^*(\alpha_{\mathcal{A}})$ is *nearly symplectic*, meaning that

$$\forall z \in S \, \forall v \in T_z S : (\Omega_{\mathcal{A}} + \pi^*(\alpha))(\widetilde{v}, \widetilde{jv}) \geq 0,$$

where $\widetilde{v}, \widetilde{jv}$ are the $\mathcal{A}$-horizontal lifts of $v, jv \in T_z S$.

Note that energy($\mathcal{A}$) could be infinite if there are no constraints on $\mathcal{A}$ at the ends.

Lemma 2.20. *Let $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be a Hamiltonian structure. For $\sigma \in \overline{\mathcal{M}}(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ we have*

$$-\int_S \sigma^* \Omega_{\mathcal{A}} \leq \text{energy}(\mathcal{A}).$$

Proof. We have

$$\int_S \sigma^*(\Omega_{\mathcal{A}} + \pi^* \alpha) \geq 0,$$

whenever $\Omega_{\mathcal{A}} + \pi^*(\alpha)$ is nearly symplectic, by the defining properties of $J_{\mathcal{A}}$ and by σ being $J_{\mathcal{A}}$-holomorphic. From which our conclusion follows. □

Lemma 2.21. *Let $\{(\widetilde{S}_t, S_t, \mathcal{L}_t, \mathcal{A}_t)\}_{[0,1]}$ be a taut concordance. Let σ_j, $j = 0, 1$ be asymptotically flat sections of $\widetilde{S}_j$ in relative class A. Then*

$$-\int_{S_1} \sigma_1^* \Omega_{\mathcal{A}_1} = -\int_{S_0} \sigma_0^* \Omega_{\mathcal{A}_0},$$

whenever both integrals are finite. In particular, for a Hamiltonian structure $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$, $\int_S \sigma^* \Omega_{\mathcal{A}}$ depends only on the relative class of A, whenever the integral is finite.

Proof. By the hypothesis, there is a connection $\mathcal{A}$ on $\widetilde{\boldsymbol{S}}$, extending each $\mathcal{A}_t$ and such that $\Omega_{\mathcal{A}}$ vanishes on $\mathbf{L} \subset \widetilde{\boldsymbol{S}}$. The first part then follows by Stokes theorem. Here are the details. For σ_j as above and for each end e_i, cut off the part of the section σ_j lying over $[0, 1] \times (t_{\delta_1, \delta_2}, \infty)$ in the corresponding end bundle chart at the end. Here t_{δ_1, δ_2} is such that $\sigma_0|_{[0,1]\times\{t\}}$ is C^1 δ_1-close to $\sigma_1|_{[0,1]\times\{t\}}$ for all $t > t_{\delta_1, \delta_2}$ and for each end, and is such that

$$\int_{[0,1]\times(t_{\delta_1,\delta_2},\infty)} \sigma_j^*|_{[0,1]\times(t_{\delta_1,\delta_2},\infty)} \Omega_{\mathcal{A}_j} < \delta_2, \; j = 1, 2,$$

for each end e_i. Call the sections with the ends cut off as above by $\sigma_j^{\delta_1,\delta_2}$, they are sections over the compact surfaces S_j^{cut}, with ends correspondingly cut off. Then by Stokes theorem, using that $\Omega_{\mathcal{A}}$ is closed and using the vanishing of $\Omega_{\mathcal{A}}$ on $\mathbf{L}$: for each ϵ there exists δ_1, δ_2 such that

$$\int_{S_1^{cut}} (\sigma_1^{\delta_1,\delta_2})^* \Omega_{\mathcal{A}} - \int_{S_0^{cut}} (\sigma_0^{\delta_1,\delta_2})^* \Omega_{\mathcal{A}} < \epsilon,$$

and

$$\int_{S_j^{cut}} (\sigma_j^{\delta_1,\delta_2})^* \Omega_{\mathcal{A}_j} - \int_{S_j} \sigma_j^* \Omega_{\mathcal{A}_j} < \epsilon, \; j = 1, 2.$$

The last part of the lemma follows from the first. For if $\mathcal{A}$ preserves $\mathcal{L}$ then $\Omega_{\mathcal{A}}$ vanishes on $\mathcal{L}$, as previously observed, and consequently the corresponding constant concordance:

$$\{(\widetilde{S}_t, S_t, \mathcal{L}_t, \mathcal{A}_t)\}_{[0,1]}, \quad \forall t \in [0,1] : (\widetilde{S}_t, S_t, \mathcal{L}_t, \mathcal{A}_t) = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$$

is taut. □

Definition 2.22. *For σ a relative class A section of $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ let us call:*

$$-\int_S \sigma^* \Omega_{\mathcal{A}},$$

*the $\mathcal{A}$-**coupling area** of σ, denoted by* $\operatorname{carea}(\Theta, \sigma)$*, we may also write* $\operatorname{carea}(\Theta, A)$ *for the same quantity. By the lemma above this is an invariant of the taut concordance class of Θ.*

Definition 2.23. *Given a Hamiltonian structure $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ we will say that Θ is A-**small** if*

$$\operatorname{energy}(\Theta) < \operatorname{carea}(\Theta, A).$$

Lemma 2.24. *Suppose that $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ is A-small then $\overline{\mathcal{M}}(\Theta, A)$ is empty. Or for a contrapositive, if $\overline{\mathcal{M}}(\Theta, A)$ is non-empty then:*

$$\operatorname{carea}(\Theta, A) \le \operatorname{energy}(\Theta).$$

Proof. This is just a reformulation of Lemma 2.20. □

2.7. **Gluing Hamiltonian structures.** Let $\mathcal{O}$ denote the Riemann surface which is topologically $D^2 - z_0$, $z_0 \in \partial D^2$, endowed with a choice of a strip end chart at the end (positive or negative depending on context). The complex structure j here is as induced from $\mathbb{C}$ under the assumed embedding $D^2 \subset \mathbb{C}$.

Let $(\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be a Hamiltonian structure. We may cap off some of the open ends $\{e_i\}_{i=0}^n$ of S, by gluing at the ends copies of $\mathcal{O}$ with oppositely signed end. More explicitly, in the strip coordinate charts at some, say positive, end e_i of S, excise $[0,1] \times (t, \infty)$ for some $t > 0$, call the resulting surface $S - e_i$. Likewise, excise the negative end of $\mathcal{O}$, call this surface $\mathcal{O} - end$. Then glue $S - e_i$ with $\mathcal{O} - end$, along their new smooth boundary components. Let us denote the capped off surface by $S^{/i}$.

Since $\widetilde{S}$ is naturally trivialized at the ends, we may similarly cap off $\widetilde{S}_r$ over the e_i end by gluing with the bundle $\mathcal{O} \times M$ at the end, obtaining a Hamiltonian M bundle $\widetilde{S}^{/i}$ over $S^{/i}$.

Moreover, we have a certain gluing operation of Hamiltonian structures. In the case of "capping off" as above we glue $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ with the Hamiltonian structure $\Theta' = (\mathcal{O} \times M, \mathcal{O}, \mathcal{L}', \mathcal{A}')$ at the e_i end, provided $\mathcal{A}'$ is compatible with the connection $\mathcal{A}_i$, in the sense of Definition 2.4, and provided $\mathcal{L}$ is compatible with $\mathcal{L}'$. The latter means that $L_i^j = {L'}_i^j$ where these are the Lagrangians as in Notation 1.

Let us name the result of this capping off $\Theta \#_i \Theta'$. The following is immediate:

Lemma 2.25. *Suppose that* $\{\Theta_r\}_{\mathcal{K}}$*,* $\{\Theta'_r\}_{\mathcal{K}}$ *with* $\Theta'_r = (\mathcal{O} \times M, \mathcal{O}, \mathcal{L}'_r, \mathcal{A}'_r)$ *are taut Hamiltonian structures. Then:*

$$\{\Theta_r \#_i \Theta'_r\}_{\mathcal{K}}$$

is taut, whenever the gluing operation is well-defined, that is whenever we have compatibility of connections and Lagrangian sub-fibrations at the corresponding end.

Definition 2.26. *Let* $\pi : \mathbb{R} \to [0,1]$ *denote the continuous retraction map, sending* $(-\infty, 0]$ *to* 0*, and sending* $[1,\infty)$ *to* 1*. Assuming the end* e_0 *of* $\mathcal{O}$ *is positive, and using the coordinates of the strip end chart* $e_0 : [0,1] \times (0,\infty) \to \mathcal{O}$*, fix the following parametrization* ζ *of the boundary of* $\mathcal{O}$*.* $\zeta : \mathbb{R} \to \partial\mathcal{O}$*, satisfies* $\zeta(t) \in \{0\} \times (0,\infty)$ *for* $t \in (-\infty, 0)$*, and* $\zeta(t) \in \{1\} \times (0,\infty)$ *for* $t \in (1,\infty)$*. Given a smooth exact path*

$$p : [0,1] \to Lag(M)$$

constant near $0, 1$*, let* $\mathcal{L}_p \subset \partial\mathcal{O} \times M$ *denote the Lagrangian subfibration over* $\partial\mathcal{O}$*, with fiber over* $r \in \partial\mathcal{O}$ *given by* $p \circ \pi(r)$*. We say that a Lagrangian subfibration* $\mathcal{L}$ *as above is* **determined by** p *if* $\mathcal{L} = \mathcal{L}_p$*, after a fixed choice of parametrization of boundary of* $\mathcal{O}$ *by* $\mathbb{R}$*. (In the case the end of* $\mathcal{O}$ *is negative, the above is meant to be analogous.)*

A Hamiltonian connection $\mathcal{A}$ on $[0,1] \times M$ uniquely corresponds to a choice of a smooth function $H : [0,1] \times M \to \mathbb{R}$, normalized to have mean zero at each moment. For the holonomy path of $\mathcal{A}$ over $[0,1]$ is a path $\phi_{\mathcal{A}} : [0,1] \to Ham(M,\omega)$, generated by a Hamiltonian $H : [0,1] \times M \to \mathbb{R}$, and this uniquely determines the connection. Conversely, H uniquely determines a Hamiltonian connection with holonomy path generated by H. We can say that H ***generates*** $\mathcal{A}$.

Lemma 2.27. *Let* p *and* $\mathcal{L}_p \subset \partial\mathcal{O} \times M$ *be as in definition above with* $L^{\pm}(\widetilde{p}) = \rho$*, where* $\widetilde{p}$ *is some lift of* p *to* $\mathrm{Ham}(M,\omega)$*, that is* $p(t) = \widetilde{p}(t)(p(0))$*. Let* $\mathcal{A}_0$ *be a Hamiltonian connection on* $[0,1] \times M$*, generated by a Hamiltonian* $H : [0,1] \times M \to \mathbb{R}$ *with* $L^{\pm}$ *length* κ*, constant for* t *near* $0, 1$*. Then there is a Hamiltonian connection* $\widetilde{\mathcal{A}}_0^p$ *on* $\mathcal{O} \times M$*, preserving* $\mathcal{L}_p$*, compatible with* $\mathcal{A}_0$*, and satisfying*

$$area(\widetilde{\mathcal{A}}_0^p) \leq \kappa + \rho.$$

The construction is natural in the sense that $(\widetilde{p}, \mathcal{A}_0) \mapsto \widetilde{\mathcal{A}}_0^p$ *can be made into a smooth map (of Frechet manifolds).*

Proof. Let $q : [0,1] \to \mathrm{Ham}(M,\omega)$ be the holonomy path of $\mathcal{A}_0$, $q(0) = id$, generated by H. Let $\widetilde{p} \cdot q$ be the usual path concatenation in diagrammatic order, and H' be its generating Hamiltonian.

Define a coupling form Ω' on $D^2 \times M$:

$$\Omega' = \omega - d(\eta(rad) \cdot H' d\theta),$$

for (rad, θ) the modified angular coordinates on D^2, $\theta \in [0,1]$, $0 \leq rad \leq 1$, and $\eta : [0,1] \to [0,1]$ is a smooth function satisfying

$$0 \leq \eta'(rad),$$

and

$$\eta(rad) = \begin{cases} 1 & \text{if } 1 - \delta \leq rad \leq 1, \\ rad^2 & \text{if } rad \leq 1 - 2\delta, \end{cases} \tag{2.28}$$

for a small $\delta > 0$. By an elementary calculation

$$\text{energy}(\mathcal{A}') = L^+(p \cdot q) = L^+(p) + L^+(q),$$

where $\mathcal{A}'$ is the connection induced by Ω'. Set

$$arc = \{(1,\theta) \in D^2 \,|\, 0 \leq \theta \leq 1/2\}.$$

Let arc^c denote the complement of arc in ∂D^2. Fix a smooth embedding $i : D^2 \hookrightarrow \mathcal{O}$ such that the following is satisfied (see Figure 1):

- The image of the embedding contains $\partial\mathcal{O} - end$, where end is the image of the distinguished (say positive) strip end chart
$$[0,1] \times (0,\infty) \to \mathcal{O}.$$
- $i(arc) \subset end^c$,
- $i(arc^c) \subset end$.

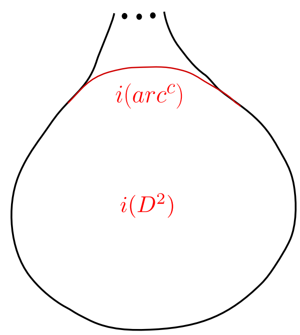

FIGURE 1.

Next fix a deformation retraction ret of $\mathcal{O}$ onto $i(D^2)$, so that in the strip end chart above, for $r \geq 1$ ret is the composition $i \circ param \circ pr$, where
$$pr : [0,1] \times (0,\infty) \to [0,1]$$
the projection and where
$$param : [0,1] \to arc^c \subset D^2$$
is a diffeomorphism. Finally, set $\Omega = ret^*\Omega'$ on $\mathcal{O} \times S^2$, and set $\widetilde{\mathcal{A}}_0^p$ to be the induced Hamiltonian connection. As constructed $\widetilde{\mathcal{A}}_0$ will be compatible with $\mathcal{A}_0$, when the end of $\mathcal{O}$ is positive. When the end is negative we take the reverse paths p^{-1}, q^{-1}. □

Let us denote by $\Theta(p, \mathcal{A}_0) = (\mathcal{O} \times M, \mathcal{O}, \mathcal{L}_p, \widetilde{\mathcal{A}}_0^p)$ the Hamiltonian structure as in the lemma above. When p is the constant map to L we will instead write
$$\Theta(L, \mathcal{A}_0). \tag{2.29}$$

The following says that under suitable conditions the connection of the lemma above can be made to have *area* 0.

Lemma 2.30. *Let $H : M \times [0,1] \to \mathbb{R}$ be a smooth time-dependent function with zero mean at each moment. Let $p : [0,1] \to \mathrm{Ham}(M,\omega)$ be the path generated by H. Let $L \in Lag(M, L_0)$, and $p_L : [0,1] \to Lag(M, L_0)$ be the path $p_L(t) = p(t)(L)$. Let $\mathcal{L}_p \in \partial\mathcal{O} \times M$ be as in the Lemma 2.27. Suppose that $\mathcal{O}$ has the positive end e_0. And let $\mathcal{A}_0$ at the e_0 be generated by H, then there is a Hamiltonian connection $\widetilde{\mathcal{A}}^H$ on $\mathcal{O} \times M$, preserving $\mathcal{L}_p$, compatible with $\mathcal{A}_0$, and satisfying*
$$area(\widetilde{\mathcal{A}}^H) = 0.$$
The construction is natural in the sense that $H \mapsto \widetilde{\mathcal{A}}^H$ can be made into a smooth map (of Frechet manifolds).

Proof. We only sketch the proof as the idea is similar to the proof of Lemma 2.27. In the notation of the proof of Lemma 2.27 let $r : i(D^2) \to i(arc)$ be a smooth retraction. Set $\mathcal{A} := r^*\mathcal{A}_0$, and set $\widetilde{\mathcal{A}}^H = ret^*\mathcal{A}$ (for ret as before). □

For future use, we denote by

$$\Theta(H) = (\mathcal{O} \times M, \mathcal{O}, \mathcal{L}_p, \widetilde{\mathcal{A}}^H), \tag{2.31}$$

the Hamiltonian structure as in the lemma above.

2.7.1. *Gluing Hamiltonian structures with estimates.* Now let $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$ be a Hamiltonian structure. For simplicity, suppose that $\mathcal{L}$ is trivial with fiber L_0, and that $\mathcal{A}$ is trivial over the boundary. Suppose further that at the end e_i the corresponding connection $\mathcal{A}_i$ is generated by $L^{\pm}$-length κ_i Hamiltonian H_i. By capping each e_i end with $\Theta(L_0, \mathcal{A}_i)$ (keeping in mind that negative-positive distinction) we obtain a Hamiltonian structure we call $\Theta^/ = (\widetilde{S}^/, S, \mathcal{L}^/, \mathcal{A}^/)$. By the lemma above:

$$\text{energy}(\mathcal{A}^/) \leq \text{energy}(\mathcal{A}) + \sum_i \kappa_i. \tag{2.32}$$

Recall that a Lagrangian $L \subset M$ is called *monotone* with monotonicity constant $const > 0$, if for any relative class $B \in H_2(M, L)$: $\omega(B) = const \cdot \mu(B)$, where μ is the Maslov number. In what follows $const$ is this monotonicity constant.

Lemma 2.33. *Let*

$$\Theta := \{\Theta_r\} := \{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}}$$

be a hyper taut Hamiltonian structure satisfying:

- *$\mathcal{L}_r$ is the trivial bundle with fiber L_0 for each r, where L_0 has monotonicity constant $const$.*
- *$\mathcal{A}_r$ is the trivial connection over the boundary of S_r for each r.*
- *The Floer chain complex $CF(\mathcal{A}_i)$ is perfect for each i and $\mathcal{A}_i$ is generated by a time dependent Hamiltonian H_i with $L^{\pm}$ length κ_i.*

Let $\Theta^/_r = (\widetilde{S}^/_r, S^/_r, \mathcal{L}^/_r, \mathcal{A}^/_r)$ be obtained from Θ_r by capping off each end e_i, so that (2.32) is satisfied. For a given $A \in H_2^{sec}(\widetilde{S}, \mathcal{L})$, if

$$\forall r : \text{energy}(\mathcal{A}_r) < \text{carea}(\Theta^/_r, A^/) - \sum_i \kappa_i,$$

where $A^/$ is the capping off of A is described in the proof, then $\overline{\mathcal{M}}(\{\Theta_r\}, A)$ is empty. Moreover,

$$\forall r : \text{carea}(\Theta^/_r, A^/) = -const \cdot Maslov^{vert}(A^/),$$

where $Maslov^{vert}$ is as in Appendix A.

Proof. Suppose otherwise that we have an element $(\sigma_0, r_0) \in \overline{\mathcal{M}}(\{\Theta_r\}, A)$. Suppose for the moment that $\overline{\mathcal{M}}(\{\Theta_r\}, A)$ is regular. There is a morphism (cf. Albers [1])

$$PSS : QH(L) \to FH(L, L),$$

where the right-hand side is defined using our construction in terms of flat sections, and the left-hand side is interpreted for example as the $\mathbb{Z}_2$ graded homology of the Pearl complex, Biran-Cornea [2]. Moreover, as shown by Albers this is an isomorphism in the present monotone context.

We won't give the full construction of this morphism in our setting, as it just a reformulation of the construction in [1]. Here is a quick sketch. Let

$$\Theta_- = (\mathcal{O} \times M, \mathcal{O}, \mathcal{L}, \mathcal{A}_-),$$

be the Hamiltonian structure with e_0 being a negative end, $\mathcal{L}$ trivial with fiber L (which is an object as before), and

$$\mathcal{A}_- := \widetilde{\mathcal{A}}_0^{p=const},$$

with right-hand side as in Lemma 2.27, for p being the constant path at L. Suppose that Θ_- is regular. Define $PSS([L])$ as the homology class of the Floer chain:

$$\sum_A ev(\Theta_-, A). \tag{2.34}$$

where the sum is over all classes $A \in H_2^{sec}(\mathcal{O} \times M, \mathcal{L})$.

Now, for a general class $a \in QH(L)$, $PSS(a)$ is defined similarly, but using the moduli space $\mathcal{M}(\Theta_-, a, A)$. The latter can be defined as the subset of $\mathcal{M}(\Theta_-, A)$ consisting of sections intersecting a fixed smooth pseudocycle, see Zinger [21], representative of a. More specifically, for $z_0 \in \partial\mathcal{O}$ let $\widetilde{S}_{z_0}$ be the fiber. Fix a pseudo-cycle $g: B \to L \subset \widetilde{S}_{z_0}$ representing $a \in H_2(L)$. Then $\mathcal{M}(\Theta_-, a, A)$ consists of elements of $\mathcal{M}(\Theta_-, A)$ intersecting image of g. (Although we use the language of pseudocycles in this outline, for analysis it is technically simpler to use Morse homology and Perl complex language as in [1].)

Now, the PSS morphism is an isomorphism in our monotone context, and $CF(\mathcal{A}_i)$ is perfect for each i, by assumption. It follows that the asymptotic constraint γ_i of σ_0 at each (positive) end e_i satisfies:

$$\langle \gamma_i, PSS(a) \rangle = 1,$$

for some a uniquely determined, where $\langle , \rangle$ is the pairing induced by the natural basis of $CF(\mathcal{A}_i)$.

Moreover, Fredholm index and monotonicity restrictions ensure that only a single class A^i can contribute in the sum. Let then $\sigma_{A^i} \in \mathcal{M}(\Theta_-, a, A_i)$ be some element.

With this understanding, at each end [3] e_i, glue σ_0 with σ_{A^i}. We then obtain a $J(\mathcal{A}'_{r_0})$-holomorphic, class A' section σ'_0 of Θ'_{r_0}.

By Lemma 2.24:

$$\text{carea}(\Theta'_{r_0}, A') \leq \text{energy}(\mathcal{A}'_{r_0}) \leq \text{energy}(\mathcal{A}_{r_0}) + \sum_i \kappa_i,$$

so

$$\text{carea}(\Theta'_{r_0}, A') - \sum_i \kappa_i \leq \text{energy}(\mathcal{A}_{r_0}),$$

so that we contradict the hypothesis. So in the case $\overline{\mathcal{M}}(\{\Theta_r\}, A)$ is regular we are done with the first part of the lemma. When it is not regular instead of gluing just pre-glue to get a holomorphic building σ'_0, and the conclusion follows by the same argument.

To prove the last part of the lemma, note that each Θ'_r is taut concordant to

$$\Theta_0 = (D^2 \times M, D^2, \mathcal{L}, \mathcal{A}^{tr}),$$

with $\mathcal{L}$ trivial with fiber L_0, and for $\mathcal{A}^{tr}$ the trivial connection. And

$$\text{carea}(\Theta_0, \cdot) = -const \cdot Maslov^{vert}(\cdot),$$

as functionals on $H_2^{sec}(D^2 \times M, \mathcal{L})$. It follows by Lemma 2.21 that

$$\text{carea}(\Theta', \sigma'_0) = \text{carea}(\Theta_0, \sigma'_0) = -const \cdot Maslov^{vert}(\sigma'_0) = -const \cdot Maslov(A').$$

□

[3] More precisely at each positive end, but in the case of negative end we do the analogous construction.

3. Quantum Maslov classes

Quantum Maslov classes are relative analogues of quantum characteristic classes in [15]. The latter give Chern classes in a certain semi-classical limit [17]. So quantum Chern classes is perhaps the most suggestive name for the construction in [15]. The name "quantum Maslov classes" is then also meant to be suggestive, as the classical Maslov numbers are relative analogues of Chern numbers.

One version of the relative Seidel morphism appears in Seidel's [20] in the exact case. This was further developed by Hu-Lalonde [7] in the monotone case.

Let $Lag(M)$ be as in the Introduction. We may also denote the component of L by $Lag(M, L)$. Then from our perspective the Hu-Lalonde morphism is a functor

$$S : \Pi Lag(M) \to DF(M),$$

where $\Pi Lag(M)$ is the category with objects elements of $Lag(M)$, with

$$hom_{\Pi Lag(M)}(L_0, L_1) = \pi_0(\mathcal{P}(L_0, L_1)),$$

where $\mathcal{P}(L_0, L_1)$ is as in the Introduction. Here, $DF(M)$ is the Donaldson-Fukaya category of M, see also [4], [3] which can be understood as an extension.

In our setup this works as follows. To an element of $\mathcal{P}(L_0, L_1)$ we have have an associated Lagrangian subbundle $\mathcal{L}_p$ of $\mathcal{O} \times M$ over the boundary, as in Definition 2.26. Extend this to a Hamiltonian structure

$$\Theta_p = (\mathcal{O} \times M, \mathcal{O}, \mathcal{L}_p, \widetilde{\mathcal{A}}_0^p)$$

where $\widetilde{\mathcal{A}}_0^p$ is as in Lemma 2.27.

Assuming Θ_p is regular, we define

$$S([p]) \in hom_{DF}(L_0, L_1) = FH(L_0, L_1)$$

by

$$S([p]) = [ev(\Theta_p)],$$

where $ev(\Theta)$ is as in Definition 2.14. Since Θ_p is well defined up to concordance, $S([p])$ is well defined by Lemma 2.16.

3.1. Definition of the quantum Maslov classes.

Let $Ob_d(X)$ denotes the d'th oriented bordism group of a space X. Set

$$Ob(\mathcal{P}(L_0, L_1)) = \sum_{d \text{ even}} Ob_d(\mathcal{P}(L_0, L_1)) \oplus \sum_{d \text{ odd}} Ob_d(\mathcal{P}(L_0, L_1)), \tag{3.1}$$

with the natural $\mathbb{Z}_2$ grading.

We construct a graded group homomorphism:

$$\Psi : Ob(\mathcal{P}(L_0, L_1)) \to FH(L_0, L_1), \tag{3.2}$$

as follows.

Let

$$f : B \to \mathcal{P}(L_0, L_1),$$

be a smooth map of a smooth connected oriented manifold B. We may suppose by homotopy approximation that each $f(s)$, as a path is constant in $[0, \epsilon] \cup [1 - \epsilon, 1]$ for some $0 < \epsilon < 1$.

We associate to this the data:

$$\{\mathcal{O} \times M, \mathcal{O}, \mathcal{L}_s\}_{s \in B},$$

$\mathcal{L}_s := \mathcal{L}_{f(s)}$ a Lagrangian subbundle of $M \times \mathcal{D}$ over $\partial \mathcal{O}$ determined by $f(s)$ as before. The end of $\mathcal{O}$ here is negative.

Now let $\mathcal{A}_0$ be a Hamiltonian connection on $[0,1] \times M$. And let $\mathcal{A}_0(L_0) \subset \{1\} \times M$ denote the $\mathcal{A}_0$-transport over $[0,1]$ of $L_0 \subset \{0\} \times M$. We suppose that $\mathcal{A}_0(L_0)$ is transverse to L_1.

Complete, the data $\{\mathcal{O} \times M, \mathcal{O}, \mathcal{L}_s\}_{s \in B}$ to a Hamiltonian structure $\{\mathcal{O} \times M, \mathcal{O}, \mathcal{L}_s, \mathcal{A}_s\}_{s \in B}$, with $\mathcal{A}_s$ compatible with $\mathcal{A}_0$ by taking a section $\{\mathcal{A}_s\}$ as in Part 5 of Definition 2.7. Then this Hamiltonian structure is well-defined up to concordance. And we suppose that $\{\mathcal{A}_s\}$ is chosen so that this is regular.

We then define $\Psi([f])$ by:

$$\Psi([f]) = [ev(\{\Theta_{f,s}\})],$$

with the right hand side as in Definition 2.14. If B is disconnected with $B = \sqcup_i B_i$, and $f : B \to \mathcal{P}(L_0, L_1)$ is smooth, we set $\Psi([f]) = \sum_i \Psi([f|_{B_i}])$. Then by Lemma 2.16 we get a well defined map:

$$\Psi : Ob(\mathcal{P}(L_0, L_1)) \to FH(L_0, L_1), \tag{3.3}$$

and by definition it is a group homomorphism. Furthermore, we have:

Lemma 3.4. *The restriction of Ψ to $\pi_d(\mathcal{P}(L_0, L_1), p_0)$ determines an additive group homomorphism:*

$$\pi_d(\mathcal{P}(L_0, L_1), p_0) \to FH(L_0, L_1)\ \forall d \geq 1,$$

where p_0 is any base point.

Proof. Let f_0, f_1 represent classes $a, b \in \pi_d(\mathcal{P}(L_0, L_1), p_0)$. And suppose that $\{\Theta_{f_0,s}\}$ is regular, then the set:

$$\mathbf{S} = \cup_A \overline{\mathcal{M}}(\Theta_{f_0,s}, A)$$

is finite, where the sum is over all A that contribute to $\Psi([f_0])$ (recall that there are only finitely many of them).

Let $F : \mathbf{S} \to S^d$ be the natural map sending an element (σ, s) to s. By finiteness of $\mathbf{S}$ we may find an $s_0 \in S^d$ such that there is a open $V_0 \ni s_0 \subset S^d$, satisfying:

$$F(\mathbf{S}) \cap U = \emptyset.$$

Let $s_1 \in S^d$ be the analogous element for the family $\{\Theta_{f_1,s}\}$ with the analogous $V_1 \ni s_1$. By doing a C^∞ small perturbation, we may suppose that the families $\{\Theta_{f_0,s}\}, \{\Theta_{f_1,s}\}$ are constant for s in open $U_0 \ni s_0$, $U_0 \subset V_0$, respectively for s in open $U_1 \ni s_1$, $U_1 \subset V_1$. Let

$$\{\Theta_{f_0 \# f_1, b}\}_{b \in S^d \# S^d}$$

denote the gluing of the two Hamiltonian structures at s_0, s_1. Here we are just doing a simple connect sum surgery, corresponding to the connect sum surgery of the spheres $S^d \# S^d \simeq S^d$ at the points s_0, s_1, with surgery restricted inside the neighborhoods U_0, U_1, see Figure 2.

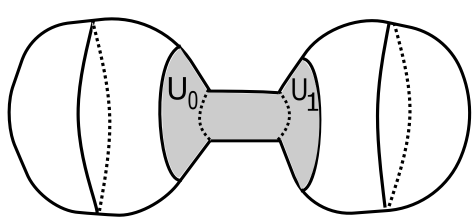

FIGURE 2. For b in the grey shaded area the connect sum family $\{\Theta_{f_0 \# f_1, b}\}$ is constant.

It is then immediate that:

$$ev(\{\Theta_{f_0 \# f_1, b}\}) = ev(\{\Theta_{f_0,s}\}) + ev(\{\Theta_{f_1,s}\}).$$

On the other hand, the structure $\{\Theta_{f_0 \# f_1,b}\}$ is clearly concordant to the structure $\{\Theta_{f_0+f_1,s}\}$, where $f_0 + f_1$ is the concatenation sum in $\pi_d(\mathcal{P}(L_0, L_1), p_0)$. So

$$ev(\{\Theta_{f_0+f_1,s}\}) = ev(\{\Theta_{f_0,s}\} + ev(\{\Theta_{f_1,s}\},$$

and the result clearly follows. □

3.2. **Multiplicative structure.** Let $C_\bullet Path(Lag(M))$ denote the A_∞ $\mathbb{Z}_2$-graded category over a commutative ring k, whose objects are elements of $Lag(M)$ and where $hom_{C_\bullet Path(Lag(M))}(L_0, L_1)$ is the $\mathbb{Z}_2$ graded k-module

$$C(\mathcal{P}(L_0, L_1), k) = \sum_{d \text{ even}} C_d(\mathcal{P}(L_0, L_1), k) \oplus \sum_{d \text{ odd}} C_d(\mathcal{P}(L_0, L_1), k).$$

The A_∞ structure comes from the classical Stasheff A_∞ composition on the chain path category.

Conjecture 2. *The morphism Ψ of Lemma 3.4 extends to a A_∞ functor*

$$\widetilde{\Psi} : C_\bullet Path(Lag(M)) \to Fuk(M),$$

where on objects $\widetilde{\Psi}(L) = L$, and on morphisms satisfying the following relation. For $a \in C(\mathcal{P}(L_0, L_1), k)$, represented by a smooth map $f : B \to \mathcal{P}(L_0, L_1)$, $[\widetilde{\Psi}(a)] = \Psi([f])$.

4. A basic computation

Our perturbation data admits Hofer theoretic functionals. The basic idea for the computation, consists of cooling the perturbation data (in the sense of the functional) to obtain a mini-max data, using which we may write down our moduli spaces explicitly.

4.1. **Hofer length.** For $p : [0, 1] \to \mathrm{Ham}(M, \omega)$ a smooth path, define

$$L^+(p) := \int_0^1 \max_M H_t^p dt,$$
$$L^-(p) := \int_0^1 \max_M (-H_t^p) dt,$$
$$L^\pm(p) := \max\{L^+(p), L^-(p)\},$$

where $H^p : M \times [0, 1] \to \mathbb{R}$ generates p, and is normalized by the condition that for each t, $H_t^p := H^p|_{M \times \{t\}}$ has mean 0, that is $\int_M H_t^p dvol_\omega = 0$.

4.1.1. *Exact paths of Lagrangians.* Recall that an exact path p of Lagrangians in M is smooth mapping $p : L \times [0, 1] \to M$, satisfying:

- $p|_{L \times \{t\}}$ is an embedding for each t, with Lagrangian image denoted $p(t)$.
- The restriction of the 1-form $p^*\omega(\frac{\partial}{\partial t}, \cdot)$ to each $L \times \{t\}$ is exact.

If $p : [0, 1] \to Lag(M)$ is an exact path, define

$$L^+_{lag} : \mathcal{P}Lag(M) \to \mathbb{R},$$
$$L^+_{lag}(p) := \int_0^1 \max_{p(t)} H_t^p dt,$$

$p(0) = L$ and where $H^p : M \times [0, 1] \to \mathbb{R}$ is normalized as above and generates a lift $\widetilde{p}$ of p to $\mathrm{Ham}(M, \omega)$ starting at id. By "lift", we mean that $p(t) = \widetilde{p}(t)(p(0))$. (That is H^p generates a path in $\mathrm{Ham}(M, \omega)$, which moves L_0 along p.) Some theory of this latter functional is developed in [8]. We may however omit the subscript lag from notation, as usually there can be no confusion which functional we mean.

4.1.2. *Restriction to standard equators in S^2.* Note that $Eq(S^2)$ is naturally diffeomorphic to S^2 and moreover it is easy to see that the functional $L^+|_{Eq(S^2)}$ is proportional to the Riemannian length functional L_{met} on the path space of S^2, with its standard round metric met. We will then sometimes identify $Eq(S^2)$ with S^2.

Let now $L_0, L_1 \in Eq(S^2)$ be any transverse pair, and

$$f' : S^2 \to \mathcal{P}^{eq}(L_0, L_1) := \{p \in \mathcal{P}(L_0, L_1) \,|\, p(t) \subset Eq(S^2),\, \forall t \in [0,1]\},$$

represent the generator a of the group $\pi_2(\mathcal{P}^{eq}(L_0, L_1), \gamma) \simeq \mathbb{Z}$, where γ denotes the unique minimal met-geodesic, from L_0 to L_1. (It is unique by the assumption that L_0, L_1 are transverse, so that the corresponding points on S^2 are not conjugate with respect to met.)

The idea of the computation is then this: perturb f' to be transverse to the (infinite dimensional) stable manifolds for the functional L_{met} on

$$\mathcal{P}^{eq}(L_0, L_1),$$

then push the cycle down by the "infinite time" negative gradient flow for this functional, and use the resulting representative to compute $\Psi(a)$. Although, we will not actually need infinite dimensional differential topology, instead we use a descent argument using Whitehead's compression lemma.

4.2. **The "energy" minimizing perturbation data.** Classical Morse theory [12] tells us that the energy functional

$$E(p) = \int_{[0,1]} \langle \dot{p}(t), \dot{p}(t) \rangle_{met}\, dt$$

on $\mathcal{P}^{eq}(L_0, L_1)$ is Morse non-degenerate with a single critical point in each degree.

Proposition 4.1. *The homotopy class a has a representative $f : S^2 \to \mathcal{P}^{eq}(L_0, L_1)$, such that:*

1. *f maps to the 2-skeleton of $\mathcal{P}^{eq}(L_0, L_1)$, for the Morse cell decomposition induced by E.*
2. *f^*E is Morse, with a single maximizer* max*, of index 2, and s.t. $\gamma_0 = f(max)$ is the index* 2 *geodesic.*

We call such a representative f **minimizing**.

Proof. This follows by Whitehead's compression lemma which is as follows.

Lemma 4.2 (Whitehead, see [6])**.** *Let (X, A) be a CW pair and let (Y, B) be any pair with $B \neq \emptyset$. For each n such that $X - A$ has cells of dimension n, assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f : (X, A) \to (Y, B)$ is homotopic relative to A to a map $X \to B$.*

Suppose that a has a representative $f' : S^2 \to \mathcal{P}^{eq}(L_0, L_1)$ mapping into the n-skeleton B^n for the Morse cell decomposition for E, $n > 2$. Apply the lemma above with $(X, A) = (S^2, pt)$, $Y = B^n$ and $B = B^{n-1}$ as above. Then the quotient B^n/B^{n-1} is a wedge of n-spheres and since $\pi_2(S^n) = 0$ for $n > 2$, f can be homotoped into B^{n-1} by the Whitehead lemma. Descend this way until we get a representative mapping into B^2.

Furthermore since $\pi_2(S^1) = 0$ such a representative cannot entirely lie in the 1-skeleton. It follows, since we have a single Morse 2-cell, that there is a representative $f : S^2 \to \mathcal{P}^{eq}(L_0, L_1)$, for a, s.t. the function f^*E is Morse with a maximizer max, of index 2, and s.t. $\gamma_0 = f(max)$ is the index 2 geodesic.

In principle there maybe more than one such maximizer max, but recall that we assumed that a is the generator, so the degree of the map $f : S^2 \to B^2/B^1 \simeq S^2$ is one. By the Hopf theorem, homotopy classes of maps of spheres are classified by degree. Hence after a further homotopy, there will be only one such maximizer. ∎

It follows that max is likewise the unique index 2 maximizer of the function f^*L_{met} by the classical relation between the energy functional and length functional. And so max is the index 2 maximizer of f^*L^+.

4.3. **The corresponding minimizing data.**

Lemma 4.3. *There is a minimizing representative f_0 for the class a and a taut Hamiltonian structure*

$$\Theta_{f_0} = \{\mathcal{O} \times S^2, \mathcal{O}, \mathcal{L}_{f_0(b)}, \mathcal{A}_b\},$$

satisfying:

(1) $f_0(b) \in Eq(S^2)$, *for each* b.

(2)

$$\forall b : \text{energy}(\mathcal{A}_b) = L^+(f_0(b)). \tag{4.4}$$

Proof. Note that a geodesic segment $p : [0,1] \to S^2 \simeq Eq(S^2)$ for the round metric met on S^2 has a unique lift

$$\widetilde{p} : [0,1] \to PU(2) \simeq SO(3),$$

$\widetilde{p}(0) = id$ with $\widetilde{p}$ a segment of a one parameter subgroup, and in this case

$$L^+_{lag}(p) = L^+(\widetilde{p}).$$

It then follows that for a piecewise geodesic path p in S^2, there is likewise a unique lift $\widetilde{p} : [0,1] \to PU(2)$, satisfying

$$L^+_{lag}(p) = L^+(\widetilde{p}).$$

Now, if f is a minimizing representative of a as above, we may homotop it to a likewise minimizing representative f_0, so that for all b $f_0(b)$ is piecewise geodesic. This follows by the piecewise geodesic approximation theorem Milnor [12, Theorem 16.2] of the loop space.

Let $\mathcal{A}_0$ be the trivial Hamiltonian connection on $[0,1] \times M$. Use the construction of Lemma 2.27, to get a family of Hamiltonian connections $\{\widetilde{\mathcal{A}}_0^{f_0(b)}\}$. In this case, since $\mathcal{A}_0$ is trivial

$$\text{energy}(\widetilde{\mathcal{A}}_0^{f_0(b)}) = L^+(f_0(b)).$$

Set $A_b = \widetilde{\mathcal{A}}_0^{f_0(b)}$. It remains to verify that $\Theta_{f_0} = \{\mathcal{O} \times S^2, \mathcal{O}, \mathcal{L}_{f_0(b)}, \mathcal{A}_b\}$ is taut, which also implies (4.4). This follows by the following lemma.

Lemma 4.5. *Any two loops $p_0, p_1 : S^1 \to Eq(S^2)$ are taut concordant as maps to $Lag(S^2)$, as defined in the Definition 1.4.*

Proof. Let $H : S^1 \times [0,1] \to Eq(S^2)$ be any smooth homotopy between p_0, p_1. Let $\mathcal{L}$ be the corresponding Lagrangian sub-fibration of $(Cyl = S^1 \times [0,1]) \times S^2$. That is for $(\theta, t) \in Cyl$, $\mathcal{L}_{(\theta,t)} = H(\theta, t)$.

Then $\mathcal{L}$ is as in Definition 1.4. Let $\mathcal{A}$ be any $PU(2)$ connection on $Cyl \times \mathbb{CP}^1$ which preserves $\mathcal{L}$ (there are no obstructions to constructing the latter.)

Then R_A is a lie algebra lie $PU(2)$ valued 2-form, such that for all $v, w \in T_z Cyl$ the vector field $X = R_{\mathcal{A}}(z)(v, w)$ is tangent to $\mathcal{L}_z$.

Let $\phi : (M_z, L_z) \to (\mathbb{CP}^1, L_0)$ be a Kahler map of the pair, where L_0 is the standard equator. Then $\phi_*(X)$ is tangent to L_0 and is in the image of

$$\mathfrak{lie}(PU(2)) \to \mathfrak{lie}(\text{Ham}(\mathbb{CP}^1, \omega_{st})).$$

In particular, the generating Hamiltonian of $\phi_*(X)$ is some standard height function on $\mathbb{CP}^1$ and so vanishes on L_0. Thus H_X vanishes on L_z. By the definition, $\Omega_{\mathcal{A}}$ vanishes on $\mathcal{L}$ and so we are done. ☐

□

So given $\{\mathcal{A}_b\}$ as in the lemma above, since

$$\forall b : \text{energy}(\mathcal{A}_b) = L^+(f_0(b)),$$

we immediately deduce:

Lemma 4.6. *The function area* $: b \mapsto \text{energy}(\mathcal{A}_b)$ *has a unique maximizer, coinciding with the maximizer* $\max$ *of* $f_0^* L_{met}$ *and* energy *is Morse at* $\max$ *with index* 2*.*

4.4. **Holomorphic sections for the data.** Let us now rename f_0 by f, $\mathcal{L}_{f_0(b)}$ by $\mathcal{L}_b$, and Θ_{f_0} by $\Theta = \{\Theta_b\}$.

As $f(\max)$ is a geodesic for met, its lift $\widetilde{f}(\max)$ to $SO(3)$ is a rotation around an axis intersecting $L_0 = f(\max)(0)$ in a pair of points, in particular there is a unique point

$$x_{\max} \in \bigcap_t (L_t = f(\max)(t))$$

maximizing $H_t^{\max}$ for each t. In our case this follows by elementary geometry but there is a more general phenomenon of this form c.f. [8].

Define

$$\sigma_{\max} : \mathcal{O} \to \mathcal{O} \times S^2$$

to be the constant section $z \mapsto x_{\max}$. Then $\sigma_{\max}$ is a $\mathcal{A}_{\max}$-flat section with boundary on $\mathcal{L}_{\max}$, and is consequently $J(\mathcal{A}_{\max})$-holomorphic.

Lemma 4.7. $\sigma_{\max}$ *satisfies:*

$$Maslov^{vert}(\sigma'_{\max}) = -2,$$

see Appendix A for the definition of this Maslov number.

Proof. Set

$$T_z^{vert} \mathcal{L}_{\max} := \{v \in T\mathcal{L} \subset T_z(\mathcal{O} \times S^2) \,|\, pr_* v = 0\}$$

where $pr : \mathcal{O} \times S^2 \to \mathcal{O}$ is the projection. Denote by

$$Lag(T_{x_{\max}} S^2 \simeq Lag(\mathbb{R}^2) \simeq S^1$$

the space of oriented linear Lagrangian subspaces of $T_{x_{\max}} S^2$. Let ρ be the path in $Lag(T_{x_{\max}} S^2)$ defined by

$$\rho(t) = T^{vert}_{(\zeta(t), x_{\max})} \mathcal{L}_{\max}, \quad t \in [0,1]$$

where $\zeta : \mathbb{R} \to \partial\mathcal{O}$ is a fixed parametrization as in Definition 2.26.

By our conventions for the Hamiltonian vector field:

$$\omega(X_H, \cdot) = -dH(\cdot),$$

ρ is a clockwise oriented path from

$$T_{x_{\max}} L_0 := T^{vert}_{(\zeta(0), x_{\max})} \mathcal{L}_{\max}$$

to

$$T_{x_{\max}} L_1 := T^{vert}_{(\zeta(1), x_{\max})} \mathcal{L}_{\max}$$

for the orientation induced by the complex orientation on $T_{x_{\max}} S^2$.

By the Morse index theorem in Riemannian geometry [12] and by the condition that $f(\max)$ has Morse index 2, ρ visits initial point $\rho(0)$ exactly twice if we count the start, as this corresponds to the geodesic $f(\max)$ passing through two conjugate points in S^2. So the concatenation of ρ with the minimal counter-clockwise path from $T_{x_{\max}} L_1$ back to $T_{x_{\max}} L_0$ is a degree -1 loop, if $S^1 \simeq Lag(\mathbb{R}^2)$ is given the counter-clockwise orientation. Consequently,

$$Maslov^{vert}(\sigma'_{\max}) = -2.$$

□

We set $A_0 = [\sigma_{\max}]$.

Proposition 4.8. $(\sigma_{\max}, \max)$ *is the sole element of* $\overline{\mathcal{M}}(\Theta, A_0)$.

Proof. By Stokes theorem, since ω vanishes on $\sigma_{\max}$, it is immediate:

$$\text{carea}(\Theta_{\max}, A_0) = -\int_{\mathcal{O}} \sigma^*_{\max} \widetilde{\Omega}_{\max} = L^+(f(\max)). \tag{4.9}$$

Moreover, since $\Theta = \{\Theta_b\}$ is taut $\text{carea}(\Theta_b, A_0) = L^+(f(\max))$. So by (4.4) and by Lemmas 2.20, 2.21 we have:

$$L^+(f(\max)) \leq \text{energy}(\mathcal{A}_b) = L^+(f(b)),$$

whenever there is an element

$$(\sigma, b) \in \overline{\mathcal{M}}(\{\Theta_b\}, A_0).$$

But clearly this is impossible unless $b = \max$, since $L^+(f(b)) < L^+(f(\max))$ for $b \neq \max$. So to finish the proof of the proposition we just need:

Lemma 4.10. *There are no elements* σ *other than* $\sigma_{\max}$ *of the moduli space*

$$\overline{\mathcal{M}}(\Theta_{\max}, A_0).$$

Proof. We have by (4.9), and by (4.4)

$$0 = \langle [\widetilde{\Omega}_{\max} + \alpha_{\widetilde{\Omega}_{\max}}], [\sigma_{\max}] \rangle,$$

and so given another element σ we have:

$$0 = \langle [\widetilde{\Omega}_{\max} + \alpha_{\widetilde{\Omega}_{\max}}], [\sigma] \rangle.$$

It follows that σ is necessarily $\widetilde{\Omega}_{\max}$-horizontal, since

$$(\widetilde{\Omega}_{\max} + \alpha_{\widetilde{\Omega}_{\max}})(v, J_{\widetilde{\Omega}_{\max}} v) \geq 0.$$

Since $J_{\widetilde{\Omega}_{\max}}$ by assumptions preserves the vertical and $\mathcal{A}_{\max}$-horizontal subspaces of $T(\mathcal{O} \times S^2)$, and since the inequality is strict for v in the vertical tangent bundle of

$$S^2 \hookrightarrow \mathcal{O} \times S^2 \to \mathcal{O},$$

the above inequality is strict whenever v is not horizontal. So σ must be $\mathcal{A}_{\max}$-horizontal. But then $\sigma = \sigma_{\max}$ since $\sigma_{\max}$ is the only flat section asymptotic to γ_0. □

□

4.4.1. *Regularity.* Since by the dimension formula (A.1) only class A_0 curves can contribute to $\Psi(a)$, it will follow that

$$\Psi(a) = \pm[\gamma_0]$$

if we knew that $(\sigma_{\max}, \max)$ was a regular element of

$$\overline{\mathcal{M}}(\{\Theta_b\}, A_0).$$

We won't answer directly if $(\sigma_{\max}, \max)$ is regular, although it likely is. But it is regular after a suitably small Hamiltonian perturbation of the family $\{\mathcal{A}_r\}$ vanishing at $\mathcal{A}_{\max}$. We call this essentially automatic regularity.

Lemma 4.11. *There is a family $\{\mathcal{A}_b^{reg}\}$ arbitrarily C^∞-close to $\{\mathcal{A}_b\}$ with $\mathcal{A}_{\max}^{reg} = \mathcal{A}_{\max}$ and such that*

$$\overline{\mathcal{M}}(\{\mathcal{O} \times S^2, \mathcal{O}, \mathcal{L}_b, \mathcal{A}_b^{reg}\}, A_0), \tag{4.12}$$

is regular, with $(\sigma_{\max}, \max)$ its sole element. In particular

$$\Psi(a) = \pm[\gamma_0].$$

Proof. The associated real linear Cauchy-Riemann operator

$$D_{\sigma_{\max}} : \Omega^0(\sigma_{\max}^* T^{vert}\mathcal{O} \times S^2_{\max}) \to \Omega^{0,1}(\sigma_{\max}^* T^{vert}\mathcal{O} \times S^2_{\max}),$$

has no kernel, by Riemann-Roch [11, Appendix C], as the vertical Maslov number of $[\sigma_{\max}]$ is -2. And the Fredholm index of $(\sigma_{\max}, \max)$ which is -2, is -1 times the Morse index of the function energy at max, by Lemma 4.6. Given this, our lemma follows by a direct translation of [18, Theorem 1.20], itself elaborating on the argument in [16]. □

To summarize:

Theorem 4.13. *For $0 \neq b \in \pi_2(\mathcal{P}^{eq}(L_0, L_1))$, and $a = i_*(b)$, where $i : \mathcal{P}^{eq}(L_0, L_1) \to \mathcal{P}(L_0, L_1)$ is the inclusion.*

$$0 \neq \Psi(a) \in HF(L_0, L_1).$$

Proof. We have shown that $0 \neq \Psi(a) \in HF(L_0, L_1)$, for a the image of the generator of the group $\pi_2(\mathcal{P}^{eq}(L_0, L_1), \mathbb{Z})$. Since Ψ is an additive group homomorphism the conclusion follows. □

Proof of the second part of Theorem 1.5 . By the theorem above the natural map

$$\pi_2(\mathcal{P}^{eq}(L_0, L_1) \to \pi_2(\mathcal{P}(L_0, L_1)),$$

is an injection and hence:

$$\pi_2(\mathcal{P}^{eq}(L_0, L_0) \to \pi_2(\mathcal{P}(L_0, L_0)),$$

is an injection and hence:

$$\pi_3(Eq(S^2)) \to \pi_3(Lag(S^2)),$$

is an injection. □

5. Proof of first part of Theorem 1.5

Suppose otherwise, so that

$$\inf_{f\in b}\max_{s\in S^2} L^+(f(s)) = U < \hbar := \frac{1}{2}\operatorname{Vol}(S^2,\omega).$$

Fix $L_1 \in Eq(S^2)$ so that L_0 intersects L_1 transversally, and so that there is a minimal geodesic path p_0 from L_0 to L_1 in $Eq(S^2)$ with

$$\kappa := L^\pm(\widetilde{p}_0) < \epsilon = (\hbar - U)/2.$$

Here $\widetilde{p}_0$ is the geodesic lift to $PU(2)$ starting at id.

Then concatenating $f(s)$ with p_0 for each s, we obtain a smooth family of paths:

$$g : S^2 \to \mathcal{P}^{eq}(L_0, L_1)$$
$$g(0) = p_0,$$

such that g represents the previously appearing class a, that is the generator of the group

$$\pi_2(\mathcal{P}(L_0,L_1),p_0).$$

Let

$$\{\Theta_s\} = \{\mathcal{O}\times S^2, \mathcal{O}, \mathcal{L}_s, \mathcal{A}_s\}_{\mathcal{K}=S^2},$$

be the corresponding Hamiltonian structure, where $\mathcal{A}_s$ is as in Lemma 4.3, defined with respect to g, and where $\mathcal{L}_s := \mathcal{L}_{g(s)}$.

In particular, $\{\Theta_s\}$ is taut and satisfies:

$$\forall s \in S^2 : \operatorname{energy}(\mathcal{A}_s) = L^+(g(s)) < \hbar - \kappa. \tag{5.1}$$

By the assumption that each $g(s)$ is taut concordant to the constant loop at L_0, each Θ_s is taut concordant to

$$\mathcal{H} = (\mathcal{O}\times S^2, \mathcal{O}, \mathcal{L}_0, \mathcal{A}),$$

where $\mathcal{L}_0 = \mathcal{L}_{p_0}$, $\mathcal{A} = \widetilde{\mathcal{A}}_0^{p_0}$, where $\widetilde{\mathcal{A}}_0^{p_0}$ is as in Lemma 2.27, for $\mathcal{A}_0$ the trivial connection.

Let $\Theta(L,\mathcal{A}_0)$ be the construction as in (2.29). Then by Lemma 2.25, for each s,

$$\Theta_s^{/0} := \Theta_s \#_0 \Theta(L_0,\mathcal{A}_0)$$

is taut concordant to

$$\mathcal{H}^{/0} := \mathcal{H}\#_0\Theta(L_0,\mathcal{A}_0).$$

On the other hand, by Lemma 4.5 $\mathcal{H}^{/0}$ is taut concordant to the trivial Hamiltonian structure $(D^2\times S^2, D^2, \mathcal{L}_{tr}, \mathcal{A}_{tr})$, where $\mathcal{L}_{tr}$ the trivial bundle with fiber L_0 and $\mathcal{A}_{tr}$ the trivial Hamiltonian connection. So for each s:

$$\operatorname{carea}(\Theta_s^{/0}, A_0) = \operatorname{carea}(\mathcal{H}^{/0}, A_0) = \hbar. \tag{5.2}$$

Now by Theorem 4.13

$$ev(\{\Theta_s\}, A_0) = \Psi(a) \neq 0.$$

And so:

$$\overline{\mathcal{M}}(\{\Theta_s\}, A_0) \neq \emptyset,$$

but this contradicts the conjunction of (5.1), (5.2), and Lemma 2.33. $\square$

Appendix A. On the Maslov number and dimension formula

Let S be a Riemann surface with boundary and a strip end structure as previously.

Let $\mathcal{V} \to S$ be a rank r complex vector bundle, trivialized at the open ends $\{e_i\}$, so that we have distinguished bundle charts $[0,1] \times (0,\infty) \times \mathbb{C}^r \to \mathcal{V}$ at the positive ends. (Similarly, for negative ends.)

Let

$$\Xi \to \partial S \subset S$$

be a totally real rank r subbundle of $\mathcal{V}$, which is constant in the coordinates

$$[0,1] \times (0,\infty) \times \mathbb{C}^r,$$

at the positive ends, again similarly with negative ends.

For each (positive) end e_i and its chart $e_i : [0,1] \times (0,\infty) \to S$, let $b_i^j : (0,\infty) \to \partial S$, $j = 0,1$ be the restrictions of e_i to $\{i\} \times (0,\infty)$.

We then have a pair of real vector spaces

$$\Xi_i^j = \lim_{\tau \mapsto \infty} \Xi|_{b_i^j(\tau)}.$$

There is a Maslov number $Maslov(\mathcal{V}, \Xi, \{\Xi_i^j\})$ associated to this data, and which we now briefly describe. In the case $\Xi_i^0 = \Xi_i^1$, let $(\mathcal{V}', \Xi')$ be obtained from $(\mathcal{V}, \Xi, \{\Xi_i^j\})$ by capping off each e_i end of $\mathcal{V} \to S$. Here the capping operation is similar to the one in Section 2.7. Then $Maslov(\mathcal{V}, \Xi, \{\Xi_i^j\})$ coincides with the boundary Maslov index of $(\mathcal{V}', \Xi')$ in the sense of [11, Appendix C3].

When Ξ_i^0 is transverse to Ξ_i^1 for each i, $Maslov(\mathcal{V}, \Xi, \{\Xi_i^j\})$ is obtained as the Maslov index for the modified pair $(\mathcal{V}', \Xi')$ obtained by again capping off the ends e_i via gluing (at each end e_i) with

$$(\mathcal{O} \times \mathbb{C}^r, \widetilde{\Xi}, \{\widetilde{\Xi}_0^j\}),$$

where $\mathcal{O}$ is as before. Here $\widetilde{\Xi}_i^0 = \Xi_0^1$ and $\widetilde{\Xi}_i^1 = \Xi_0^0$, while $\widetilde{\Xi}$ over the boundary of $\mathcal{O}$ is determined by the "shortest path" from $\widetilde{\Xi}_0^0$ to $\widetilde{\Xi}_0^1$, which means the following. As $\widetilde{\Xi}_0^0$ to $\widetilde{\Xi}_0^1$ are a pair of transverse, totally real subspaces, up to a complex isomorphism of $\mathbb{C}^r$ (whose choice will not matter), we may identify them with the subspaces $\mathbb{R}^r$, and $i\mathbb{R}^r$. After this identification our shortest path is just $e^{i\theta}\mathbb{R}^r$, $\theta \in [0, \pi_2]$.

Let D be a real linear Cauchy-Riemann operator on $\mathcal{V}$, which in particular is an operator:

$$D : \Omega^0_\Xi(S, \mathcal{V}) \to \Omega^{0,1}_\Xi(S, \mathcal{V}),$$

where $\Omega^0_\Xi(S, \mathcal{V})$ denotes the space of smooth $\mathcal{V}$-valued 0-forms (i.e. sections) satisfying $\theta(\partial S) \subset \Xi$, and $\Omega^{0,1}_\Xi(S, \mathcal{V})$ denotes the analogous space of smooth complex anti-linear 1-forms.

Suppose further that D is asymptotically $\mathbb{R}$-invariant in the end bundle charts. After standard Sobolev completions, the Fredholm index of D is given by:

$$r \cdot \chi(S) + Maslov(\mathcal{V}, \Xi, \{\Xi_i\}).$$

The proof of this is analogous to [11, Appendix C], we can also reduce it to that statement via a gluing argument. (This kind of argument appears for instance in [20])

A.1. **Dimension formula for moduli space of sections.** Suppose we have a Hamiltonian structure $\Theta = (\widetilde{S}, S, \mathcal{L}, \mathcal{A})$. Suppose that either the corresponding Lagrangian submanifolds

$$L_i^j = \lim_{\tau \mapsto \infty} \mathcal{L}|_{b_i^j(\tau)},$$

intersect transversally (identifying the corresponding fibers) or coincide. (Similarly for negative ends.)

Let $A \in H_2^{sec}(\widetilde{S}, \mathcal{L})$, with the latter as in Section 2.3. And let $\mathcal{M}(\Theta, A)$ be as in Section 2.5. Define

$$Maslov^{vert}(A)$$

to be the Maslov number of the triple $(\mathcal{V}, \Xi, \{\Xi_i\})$ determined by the pullback by $\sigma \in \mathcal{M}(A)$ of the vertical tangent bundle of $\widetilde{S}$, $\mathcal{L}$. Then the expected dimension of $\mathcal{M}(A)$ is:

$$r \cdot \chi(S) + Maslov^{vert}(A). \tag{A.1}$$

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