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SMOOTH SIMPLICIAL SETS AND UNIVERSAL CHERN-WEIL FOR INFINITE DIMENSIONAL GROUPS

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ABSTRACT. We give the construction of the universal, natural up to homotopy Chern-Weil differential graded algebra homomorphism:

$$cw : \mathcal{I}(G) \to \Omega^\bullet(BG, \mathbb{R})$$

for infinite dimensional Milnor regular Lie groups G, where $\Omega^\bullet(BG, \mathbb{R})$ is a certain de Rham algebra of BG (Milnor BG up to a natural weak homotopy equivalence) and where $\mathcal{I}(G)$ is the algebra of continuous, Ad_G invariant, symmetric multilinear functionals on the Lie algebra. In particular, this applies to the group of compactly generated Hamiltonian symplectomorphisms, using which we verify a conjecture of Reznikov. For the construction of cw we introduce a basic geometric-categorical notion of a smooth simplicial set. Loosely, this is to Chen spaces as simplicial sets are to spaces. We then give a new construction of the classifying space of G as a smooth Kan complex, with the geometric realization weakly equivalent to the Milnor BG.

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1. Introduction

First, we introduce the notion of a smooth simplicial set, which is most directly an analogue in the world of simplicial sets of Chen spaces [3], and less directly of diffeological spaces of Souriau [44]. The Chen/diffeological spaces are perhaps the most basic notions of a "smooth space". The language of smooth simplicial sets turns out to be a powerful tool to resolve the long standing problem of the construction of the universal Chern-Weil *dg* algebra homomorphism for infinite dimensional Lie groups.

1.1. **Generalized Lie groups.** By an infinite dimensional Lie group G we will mean a Lie group whose underlying infinite dimensional smooth manifold is modeled on a locally convex topological vector space. For Chern-Weil theory we will need that G is Milnor regular, see [34, Definition II.5.2]. This condition may not ultimately be necessary, see Remark 7.1. As explained in [34], a basic example of a regular group is the group $\mathrm{Diff}_c(M)$ of compactly supported diffeomorphisms of a smooth finite dimensional manifold. In this case regularity boils down to the fact that a smooth family of smooth (compactly supported) vector fields $\{X_t\}$ integrates to a smooth family of self-diffeomorphisms (a.k.a. a smooth isotopy), and the dependence of this on $\{X_t\}$ is itself smooth. [1] The group $\mathrm{Diff}_c(M)$ in addition has the homotopy type of a CW complex, see [14]. This also applies to other LF (limit Frechet) generalized Lie groups like the group of compactly generated Hamiltonian symplectomorphisms, which forms an important example for us.

Since this definition also encompasses standard finite dimensional Lie groups, it is convenient to give this a new working name: ***generalized Lie group***.

1.2. **Universal *dg* Chern-Weil homomorphism.** One problem of topology is the construction of a "smooth structure" on the Milnor classifying space BG of a generalized (real or complex) Lie group G. There are specific requirements for what such a notion of a smooth structure should entail. At the very least we hope to be able to carry out Chern-Weil theory universally on BG. We now describe this.

In what follows, we keep to real generalized Lie groups G, but there is no essential difficulty to extending our theory to the complex case. Denote by $\mathcal{I}(G)$ the algebra over $\mathbb{R}$ of Ad_G invariant, continuous, symmetric multilinear functionals on the Lie algebra $\mathfrak{g}$, see Section 7.1.2.

Denote by $\Omega^\bullet(BG, \mathbb{R})$ the "*dg* algebra of differential forms on BG" (for the moment left unspecified) whose cohomology is $H^\bullet(BG, \mathbb{R})$. Then we want a "purely" differential geometric construction of the Chern-Weil *dg* algebra homomorphism (natural up to homotopy of *dg* maps, Definition 7.3):

$$cw : \mathcal{I}(G) \to \Omega^\bullet(BG, \mathbb{R}),$$

where the differential on the left is trivial (and the grading is in even degrees). The goal is to set up all structures in such a way that the differential geometry in this construction becomes trivial, modulo the inherent differential geometry involved in Chern-Weil forms.

For finite dimensional Lie groups, the cohomological universal Chern-Weil homomorphism:

$$hcw : \mathcal{I}(G) \to H^\bullet(BG, \mathbb{R}) \tag{1.1}$$

has been studied for instance by Bott [1]. It has also been directly constructed in Dupont [7] using simplicial techniques, also important for us here. However, to

[1]As I understand, there are no known examples of locally convex Lie groups that are not regular, see however [28] for a diffeological example.

produce a (natural) *dg* enhancement of this, some geometric theory is ostensibly required. For example, working on the *dg* level is important in Freed-Hopkins [9]. However they work on smooth classifying stacks (and with classical Lie groups G), rather then on the Milnor spaces BG.

1.2.1. *Smooth structures on BG.* One candidate for a smooth structure on BG is some kind of diffeology. For example Magnot and Watts [29] construct a natural diffeology on the Milnor classifying space BG. Another approach to this is contained in Christensen-Wu [5], where the authors also state their plan to develop some kind of universal Chern-Weil theory in the future.

A further specific possible requirement for the above discussed "smooth structures", is that the smooth singular simplicial set $BG_\bullet$ should have a geometric realization weakly homotopy equivalent to BG. See for instance [23] for one approach to this particular problem in the context of diffeologies. In the category of smooth simplicial sets we have a stronger analogue of this, in the form of Proposition 3.7. The latter and more specifically Theorem 8.6 is used in [40] to prove universality of the global Fukaya category.

The structure of a smooth simplicial set is initially more flexible than a space with diffeology, but we may add further conditions, like the Kan condition, which will be important for us. Given a generalized Lie group G, for each choice of a particular kind of Grothendieck universe $\mathcal{U}$ we construct a smooth simplicial set $BG^{\mathcal{U}}$ with a specific classifying property. We note that this is *not* the Milnor construction. But it will be shown that $|BG^{\mathcal{U}}|$ always has the weak homotopy type of Milnor BG.

The simplicial set $BG^{\mathcal{U}}$ is moreover a Kan complex, and so is a basic example of what we call a smooth Kan complex. Our constructions, will work naturally on $BG^{\mathcal{U}}$ rather than its geometric realization. And all the desires of "smoothness" mentioned above then is some sense hold true for $BG^{\mathcal{U}}$ via its smooth Kan complex structure.

1.2.2. *Cohomological Chern-Weil homomorphism.* We first note that the cohomological Chern-Weil homomorphism 1.1 has a direct extension for generalized Lie groups. The base space Y of ordinary smooth G-bundles [2] is throughout assumed to be a finite dimensional smooth manifold, without corners unless specified so. The following is proved in Section 9.1.

Theorem 1.1. *Let G be a generalized Lie group then there is an algebra homomorphism:*

$$hcw : \mathcal{I}(G) \to H^\bullet(BG, \mathbb{R}).$$

This has the following property. Suppose that $P \to Y$ is a smooth G-bundle over a smooth manifold Y, and

$$hcw^P : \mathcal{I}(G) \to H^\bullet(Y, \mathbb{R})$$

is the associated Chern-Weil map. Then

$$hcw^P = f_P^* \circ hcw$$

[2] A G-bundle is henceforth a short name for: a principal G bundle.

where

$$f_P : Y \to BG$$

is the classifying map of P and

$$f_P^* : H^\bullet(BG, \mathbb{R}) \to H^\bullet(Y, \mathbb{R})$$

the induced algebra map.

1.3. *dg* **Enhancement.** In the infinite dimensional case, the *dg* algebra $\mathcal{I}(G)$ may not be freely generated [3]. Therefore we cannot readily enhance the map *hcw* to a differential graded map purely formally. Furthermore, even when a formal enhancement is possible it may not be adequate for more in depth geometric applications like the theory of Cheeger-Simons characters, [2], which needs suitably natural differential forms.

We can at first avoid Grothendieck universes in the statement, and in fact get a purely algebraic-topological result. We will however need the notion of geometric homotopy of *dg* maps [4] as given in Definition 7.3, the latter is shown in the Appendix A to induce an A_∞ homotopy of *dg* maps. However, unless specified otherwise homotopy will mean geometric homotopy throughout. Let $A(BG)$ denote the Whitney-Sullivan commutative *dga* of BG over $\mathbb{R}$, as reviewed in Section 9.3. The following is also proved there:

Theorem 1.2. *Let G be a generalized Lie group having the homotopy of a CW complex (e.g. $G = \mathrm{Diff}_c(M)$). Then there is a lifting:*

$$\begin{array}{ccc} & & A(BG) \\ & \overset{cw}{\nearrow} & \downarrow \\ \mathcal{I}(G) & \xrightarrow{hcw} & H^\bullet(BG, \mathbb{R}), \end{array}$$

where the arrow on the right is the cohomology projection, and cw is a dg map that is natural up to homotopy.

Remark 1.3. *As one ingredient the proof uses the Grothendieck's axiom of universes, as there are no universes in the statement one may wonder if there is a pure ZFC proof of the above.*

There is an ostensibly stronger formulation of the above. More specifically, once a "smooth model" for BG as a smooth Kan complex $BG^{\mathcal{U}}$ is fixed, there is a canonical "de Rham algebra" $\Omega^\bullet(BG^{\mathcal{U}}, \mathbb{R})$, as described in Section 4. We also need some additional ingredients. If $P \to Y$ is a $\mathcal{U}$-small G-bundle, then there is a certain classifying simplicial map $f_{P^\Delta} : Y_\bullet \to BG^{\mathcal{U}}$, see Theorem 8.6, where $Y_\bullet$ denotes the smooth singular set of Y. Also there is an obvious natural *dg* map:

$$\Theta : \Omega^\bullet(Y, \mathbb{R}) \to \Omega^\bullet(Y_\bullet, \mathbb{R}).$$

The following is proved in Section 9.2.

[3] This was nicely pointed out to me by a referee, using Reznikov's tensors (appearing just ahead) as an example.

[4] A *dg* map will be shorthand for *dg* algebra homomorphism.

Theorem 1.4. *Let G be a generalized Lie group and $\mathcal{U}$ a G-admissible Grothendieck universe. For each choice of a G-connection D on the universal G-bundle $EG^{\mathcal{U}} \to BG^{\mathcal{U}}$ there is a natural dg map:*

$$cw^D : \mathcal{I}(G) \to \Omega^\bullet(BG^{\mathcal{U}}, \mathbb{R}),$$

whose homotopy class is independent of the choice of D. The map has the following property. Suppose that $P \to Y$ is a $\mathcal{U}$-small smooth G-bundle, over a smooth manifold Y, and

$$cw^P : \mathcal{I}(G) \to \Omega^\bullet(Y, \mathbb{R})$$

is the associated standard Chern-Weil dg map (natural up to homotopy). Then

$$\Theta \circ cw^P \simeq f^*_{P^\Delta} \circ cw, \quad (\textit{homotopy relation}).$$

1.4. An example: the group of Hamiltonian symplectomorphisms.

Here is one concrete example, with more details in Sections 10 and 11. Let $\mathcal{H} = \mathrm{Ham}(M, \omega)$ denote the generalized Lie group of compactly generated Hamiltonian symplectomorphisms of some symplectic $2n$-manifold (M, ω). Here ϕ is compactly generated means that there is a smooth compactly supported $H : M \times [0, 1] \to \mathbb{R}$ s.t. ϕ is the time one map of the Hamiltonian flow $\{X_t\}$, $\omega(X_t, \cdot) = dH_t$.

Let $\mathfrak{h}$ denote the Lie algebra of $\mathcal{H}$. When M is compact $\mathfrak{h}$ is naturally isomorphic to the space of mean 0 (with respect to $dvol_\omega = \omega^n$) smooth functions on M; otherwise it is the space of all smooth compactly supported functions. In [37] Reznikov defined $Ad_{\mathcal{H}}$ invariant, continuous, symmetric multilinear functionals $\{r_k\}_{k \geq 1}$ on the Lie algebra $\mathfrak{h}$. These are defined by:

$$(H_1, \ldots, H_k) \mapsto \int_M H_1 \cdot \ldots \cdot H_k \, \omega^n.$$

Denote by $\mathcal{R}ez$ the sub-algebra of $\mathcal{I}(\mathcal{H})$ generated by $\{r_k\}$.

By Chern-Weil theory we get cohomology classes $c^{r_k}(P) \in H^{2k}(X, \mathbb{R})$ for any smooth $\mathcal{H}$-bundle P over a smooth manifold X. The following results are proved in Section 10. Using Theorem 1.4 we get:

Corollary 1.5. *There is a dg map (natural up to homotopy)*

$$rez : \mathcal{R}ez \to \Omega^\bullet(B\mathcal{H}^{\mathcal{U}}, \mathbb{R}),$$

satisfying the restriction property as in Theorem 1.4. And in particular, there are universal Reznikov cohomology classes $c^{r_k} \in H^{2k}(B\mathcal{H}, \mathbb{R})$, satisfying the following. Let $Z \to Y$ be a smooth principal $\mathcal{H}$-bundle. Let $c^{r_k}(Z) \in H^{2k}(Y)$ denote the Reznikov class. Then

$$f_Z^* c^{r_k} = c^{r_k}(Z),$$

where $f_Z : Y \to B\mathcal{H}$ is the classifying map of the underlying topological $\mathcal{H}$-bundle.

The second part of the corollary is an explicit form of a statement asserted by Reznikov [37, page 12], on the extension of his classes to the universal level on $B\mathcal{H}$ [5]. The above corollary is actually stronger, since we don't require compactness of M.

[5] His assertion is left without proof.

Likewise, we obtain a differential geometric proof that the Guillemin-Sternberg-Lerman coupling class $\mathfrak{c}(P) \in H^2(P)$ [10], [30] of a Hamiltonian fibration (Definition 11.1) has a universal representative. Specifically, let $M^{\mathcal{H}}$ denote the M-fibration associated to the universal principal $\mathcal{H}$-fibration $\mathcal{E} \to B\mathcal{H}$. (In other words the universal Hamiltonian M-bundle.)

Theorem 1.6. *There is a cohomology class $\mathfrak{c} \in H^2(M^{\mathcal{H}})$ so that if $P \to X$ is a smooth Hamiltonian M-fibration and $\widetilde{f} : P \to M^{\mathcal{H}}$ the corresponding map then $\widetilde{f}_P^* \mathfrak{c} = \mathfrak{c}(P)$.*

For M closed this is proved by Kedra-McDuff [18, Proposition 3.1] using homotopy theory techniques.

Here is a basic application. Let $Symp(\mathbb{CP}^k)$ denote the group of symplectomorphisms of $\mathbb{CP}^k$, that is diffeomorphisms $\phi : \mathbb{CP}^k \to \mathbb{CP}^k$ s.t. $\phi^* \omega_0 = \omega_0$ for ω_0 the Fubini-Study symplectic 2-form on $\mathbb{CP}^k$. Using the above corollary, we may obtain an elementary proof of the following theorem of Kedra-McDuff:

Theorem 1.7 (Kedra-McDuff)**.** *Let*

$$i : BPU(n) \to BSymp(\mathbb{CP}^{n-1})$$

be the natural map. Then

$$i_* : H_\bullet(BPU(n), \mathbb{R}) \to H_\bullet(BSymp(\mathbb{CP}^{n-1}), \mathbb{R})$$

is an injection for all $n \geq 2$.

More history and background surrounding these theorems is in Sections 9 and 10.

1.5. **Other examples.** One other basic set of examples of generalized Lie groups, with a wealth of invariant polynomials on the Lie algebra, are the loop groups. That is the groups $LG, \Omega G$, where G is any (finite dimensional) Lie group, LG is the free loop space, and ΩG is the based loop space at id. See for instance [35] for related computations. Loop groups are prominent in conformal field theory, see for instance [36] for the foundation of the subject. The relevant Chern-Weil theory then has physical connotations. Other examples of infinite dimensional Chern-Weil theory include: [26], [31], [38], [25].

There are various precedents in giving a differential geometric definition of the (infinite dimensional group) Chern-Weil homomorphism in some cases, for example Magnot [27].

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2. Preliminaries and notation

We denote by Δ the simplex category:

- The set of objects of Δ is $\mathbb{N}$.
- $\hom_\Delta(n, m)$ is the set of non-decreasing maps $[n] \to [m]$, where $[n] = \{0, 1, \ldots, n\}$, with its natural order.

A simplicial set X is a functor

$$X : \Delta^{op} \to Set.$$

The set $X(n)$ is called the set of n-simplices of X. Given a collection of sets $\{X(n)\}_{n \in \mathbb{N}}$, by a ***simplicial structure*** we will mean the extension of this data to a functor: $X : \Delta^{op} \to Set$.

Δ^d_{simp} will denote a particular simplicial set: the standard representable d-simplex, with

$$\Delta^d_{simp}(n) = hom_\Delta(n, d).$$

A morphism or a map of simplicial sets, or a ***simplicial map*** $f : X \to Y$ is a natural transformation f of the corresponding functors. The category of simplicial sets will be denoted by $s - Set$.

By a d-simplex Σ of a simplicial set X, we may mean, interchangeably, either the element in $X(d)$ or the map of simplicial sets:

$$\Sigma : \Delta^d_{simp} \to X,$$

uniquely corresponding to Σ via the Yoneda lemma. If we write Σ^d for a simplex of X, it is implied that it is a d-simplex.

With the above identification if $f : X \to Y$ is a map of simplicial sets then

$$f(\Sigma) = f \circ \Sigma. \tag{2.1}$$

2.1. Topological simplices and smooth singular simplicial sets.

Let Δ^d be the topological d-simplex, i.e.

$$\Delta^d := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \,|\, x_1 + \ldots + x_d \leq 1, \text{ and } \forall i : x_i \geq 0\}.$$

The vertices of Δ^d will be assumed ordered in the standard way.

Definition 2.1. *Let X be a smooth manifold with corners, in the diffeological sense* [11]. *We say that a map $\sigma : \Delta^n \to X$ is smooth if it is smooth as a map of manifolds with corners. In particular, $\sigma : \Delta^n \to \Delta^d$ is smooth iff it has an extension to a smooth map $V \supset \Delta^n \to \mathbb{R}^d$, with V open. (See* [17, Theorem 4.1]*).*

We denote by $\Delta^d_\bullet$ the smooth singular simplicial set of Δ^d, i.e. $\Delta^d_\bullet(k)$ is the set of smooth maps

$$\sigma : \Delta^k \to \Delta^d.$$

We call an affine map $\Delta^k \to \Delta^d$ taking vertices to vertices, in an order preserving way, ***simplicial***. And we denote by

$$\Delta^d_{simp} \subset \Delta^d_\bullet$$

the sub-simplicial set consisting of these topological simplicial maps. That is $\Delta^d_{simp}(k)$ is the set of simplicial maps $\Delta^k \to \Delta^d$.

Note that Δ^d_{simp} is naturally isomorphic to the standard representable d-simplex Δ^d_{simp} as previously defined, so that this abuse of notation should not cause issues. Thus we may also understand Δ as the category with objects topological simplices Δ^d, $d \geq 0$ and morphisms simplicial maps.

Notation 2.2. *A morphism $m \in hom_{\Delta}(n,k)$ uniquely corresponds to a simplicial map $\Delta^n_{simp} \to \Delta^k_{simp}$, which uniquely corresponds to a topological simplicial map $\Delta^n \to \Delta^k$ (as defined right above). The correspondence is by taking the maps $\Delta^n_{simp} \to \Delta^k_{simp}$, $\Delta^n \to \Delta^k$, to be determined by the set map $m : \{0,\ldots,n\} \to \{0,\ldots,k\}$. We will not notationally distinguish these corresponding morphisms. So that m may simultaneously refer to all of the above morphisms.*

2.2. **The simplex category of a simplicial set.**

Definition 2.3. *For X a simplicial set, $\Delta(X)$ will denote a certain category called the* **simplex category of** X*. This is the category s.t.:*

- *The set of objects* $\operatorname{obj}\Delta(X)$ *is the set of simplices*
$$\Sigma : \Delta^d_{simp} \to X, \quad d \geq 0.$$
- *Morphisms $f : \Sigma_1 \to \Sigma_2$ are commutative diagrams in $s-Set$:*

$$\begin{array}{ccc} \Delta^d_{simp} & \xrightarrow{\widetilde{f}} & \Delta^n_{simp} \\ & \searrow^{\Sigma_1} & \downarrow{\scriptstyle \Sigma_2} \\ & & X \end{array} \tag{2.2}$$

with top arrow a simplicial map, which we denote by $\widetilde{f}$.

An object $\Sigma : \Delta^d_{simp} \to X$ is likewise called a d-simplex, and such a Σ will be said to have degree d. We may specify the degree with a superscript, for example Σ^d for degree d.

Definition 2.4. *We say that $\Sigma^m \in \Delta(X)$ is* **non-degenerate** *if there is no morphism*

$$\begin{array}{ccc} \Delta^m_{simp} & \xrightarrow{\widetilde{f}} & \Delta^n_{simp} \\ & \searrow^{\Sigma^m} & \downarrow{\scriptstyle \Sigma^n} \\ & & X \end{array}$$

s.t. $m > n$.

There is a forgetful functor
$$T : \Delta(X) \to \Delta,$$
$T(\Sigma^d) = \Delta^d_{simp}$, $T(f) = \widetilde{f}$. We denote by $\Delta^{inj}(X) \subset \Delta(X)$ the sub-category with same objects, and morphisms f such that $\widetilde{f}$ are monomorphisms, i.e. are face inclusions.

2.3. **Geometric realization.** Let Top be the category of topological spaces. Let X be a simplicial set, then define as usual the ***geometric realization*** of X by the colimit in Top:

$$|X| := \operatorname{colim}_{\Delta(X)} T,$$

for $T : \Delta(X) \to \Delta \subset Top$ as above, understanding Δ as a subcategory of Top as previously explained.

3. Smooth simplicial sets

If

$$\sigma : \Delta^d \to \Delta^n$$

is a smooth map then we have an induced map of simplicial sets

$$\sigma_\bullet : \Delta^d_\bullet \to \Delta^n_\bullet, \tag{3.1}$$

defined by

$$\sigma_\bullet(\rho) = \sigma \circ \rho.$$

We now give a pair of equivalent definitions of smooth simplicial sets. The first is more hands on, and has a close connection to the definition of Chen/diffeological spaces, while the second is more conceptual/categorical.

Definition 3.1 (First definition)**.** *A* **smooth simplicial set** *consists of the following data:*

(1) *A simplicial set X.*

(2) *For each $\Sigma : \Delta^n_{simp} \to X$ an n-simplex there is an assigned map of simplicial sets*

$$g(\Sigma) : \Delta^n_\bullet \to X.$$

This satisfies:

(a)

$$g(\Sigma)|_{\Delta^n_{simp}} = \Sigma. \tag{3.2}$$

We abbreviate $g(\Sigma)$ by Σ_*, when there is no need to disambiguate which structure g is meant.

(b) *The following property will be called* **push-forward functoriality***:*

$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_\bullet \tag{3.3}$$

where $\sigma : \Delta^k \to \Delta^d$ is a k-simplex of $\Delta^d_\bullet$, and where Σ as before is a d-simplex of X.

Thus, formally a smooth simplicial set is a 2-tuple (X, g), satisfying the axioms above. When there is no need to disambiguate we omit specifying g.

Definition 3.2. *A* **smooth** *map between smooth simplicial sets*

$$(X_1, g_1), (X_2, g_2)$$

is a simplicial map

$$f : X_1 \to X_2,$$

which satisfies the condition:

$$\forall n \in \mathbb{N}\, \forall \Sigma \in X_1(n) : g_2(f(\Sigma)) = f \circ g_1(\Sigma), \tag{3.4}$$

or more succinctly:

$$\forall n \in \mathbb{N}\, \forall \Sigma \in X_1(n) : (f(\Sigma))_* = f \circ \Sigma_*.$$

A ***diffeomorphism*** between smooth simplicial sets is defined to be a smooth map, with a smooth inverse.

Now let Δ^{sm} denote the category:

(1) The set of objects of Δ^{sm} is $\mathbb{N}$.

(2) $\hom_{\Delta^{sm}}(k, n)$ is the set of smooth maps $\Delta^k \to \Delta^n$.

(3) The composition of morphism is the natural composition.

Definition 3.3 (Second definition)**.** *A* **smooth simplicial set** *X is a functor $X : (\Delta^{sm})^{op} \to Set$. A smooth map $f : X \to Y$ of smooth simplicial sets is defined to be a natural transformation from the functor X to Y.*

The equivalence of the above definitions is established further ahead, as we need certain preliminaries. In what follows, we refer to the first definition unless specified otherwise.

Remark 3.4. *There are respective advantages to both definitions. With the second definition we can lean more on category theory. In particular, some of the technical results ahead are incarnations of the Yoneda lemma and other such tools. For the first definition one can work with the Kan condition more directly, and it is simpler to relate the first definition to the existing theory of diffeological/Chen spaces, and to the existing theory of differential forms on simplicial sets.*

Example 3.5 (The tautological smooth simplicial set)**.** *$\Delta^n_\bullet$ has a tautological smooth simplicial set structure, where*

$$g(\Sigma) = \Sigma_\bullet,$$

for $\Sigma : \Delta^k \to \Delta^n$ a smooth map, hence a k-simplex of $\Delta^n_\bullet$, and where $\Sigma_\bullet$ is as in (3.1).

Lemma 3.6. *Let X be a smooth simplicial set and $\Sigma : \Delta^n_{simp} \to X$ an n-simplex. Let $\Sigma_* : \Delta^n_\bullet \to X$ be the corresponding simplicial map. Then Σ_* is smooth with respect to the tautological smooth simplicial set structure on $\Delta^n_\bullet$ as above.*

Proof. Let σ be a k-simplex of $\Delta^n_\bullet$, so $\sigma : \Delta^k \to \Delta^n$ is a smooth map. We need that

$$(\Sigma_*(\sigma))_* = \Sigma_* \circ \sigma_*.$$

Now $\sigma_* = \sigma_\bullet$, by definition of the tautological smooth structure on $\Delta^n_\bullet$. So we have:

$$\begin{aligned}(\Sigma_*(\sigma))_* &= \Sigma_* \circ \sigma_\bullet \text{ by Axiom 2b} \\ &= \Sigma_* \circ \sigma_*.\end{aligned}$$

∎

The following proposition in particular tells us that the weak homotopy type of a smooth simplicial set (as a plain simplicial set) is determined by its complex of smooth simplices. For contrast, the weak homotopy type of Chen/Diffeological space is not generally determined by its complex of smooth simplices (cf. [4, Example 3.12]).

Proposition 3.7. *The set (just set) of n-simplices of a smooth simplicial set X is naturally isomorphic to the set of smooth maps $\Delta^n_\bullet \to X$. In fact, define $X_\bullet$ to be the simplicial set whose n-simplices are smooth maps $\Delta^n_\bullet \to X$, and so that if $i: m \to n$ is a morphism in Δ then*

$$X_\bullet(i): X(n) \to X(m)$$

is the "pull-back" map:

$$X_\bullet(i)(\Sigma) = \Sigma \circ i_\bullet,$$

for $i_\bullet : \Delta^m_\bullet \to \Delta^n_\bullet$ the induced map. Then $X_\bullet$ is naturally isomorphic to X.

Proof. Let $\rho : \Delta^n_{simp} \to X$ be an n-simplex. By the lemma above, we have a uniquely associated to it smooth map $\rho_* : \Delta^n_\bullet \to X$. Conversely, suppose we are given a smooth map $m : \Delta^n_\bullet \to X$. Then we get an n-simplex $\rho_m := m|_{\Delta^n_{simp}}$. Let $id^n : \Delta^n \to \Delta^n$ be the identity map. We have that

$$\begin{aligned} m = m \circ id^n_\bullet &= m \circ id^n_* \\ &= (m(id^n))_*, \text{ as } m \text{ is smooth} \\ &= (\rho_m(id^n))_*, \text{ trivially by definition of } \rho_m \\ &= (\rho_m)_* \circ id^n_*, \text{ as } (\rho_m)_* \text{ is smooth by Lemma 3.6} \\ &= (\rho_m)_*. \end{aligned}$$

Thus, the map $I_n(\rho) = \rho_*$, from the set of n-simplices of X to the set of smooth maps $\Delta^n_\bullet \to X$, is bijective.

Given an element $m \in hom_\Delta(n, d)$, let $m_{simp} : \Delta^n_{simp} \to \Delta^d_{simp}$ denote the corresponding natural transformation, also identified with an element of $\Delta^d_{simp}(n)$. Then the corresponding map

$$X(m) : X(d) \to X(n)$$

is

$$\rho \mapsto \rho \circ m_{simp},$$

for $\rho : \Delta^n_{simp} \to X$.

With that in mind, the diagram below commutes

$$\begin{array}{ccc} X(d) & \xrightarrow{X(m)} & X(n) \\ \downarrow{I_d} & & \downarrow{I_n} \\ X_\bullet(d) & \xrightarrow{X_\bullet(m)} & X_\bullet(n), \end{array}$$

as

$$\begin{aligned} X_\bullet(m) \circ I_d(\rho) &= X_\bullet(m)(\rho_*) \\ &= \rho_* \circ m_\bullet \end{aligned}$$

while

$$\begin{aligned} I_n \circ X(m)(\rho) &= (\rho \circ m_{simp})_* \\ &= (\rho_* \circ m_{simp})_*, \text{ by (3.2)} \\ &= (\rho_*(m_{simp}))_*, \text{ by (2.1)} \\ &= \rho_* \circ m_\bullet, \text{ by Axiom 3.3.} \end{aligned}$$

Thus I is a natural transformation and is an isomorphism of simplicial sets $I : X \to X_\bullet$. □

Lemma 3.8. *Given a smooth* $m : \Delta^d_\bullet \to \Delta^n_\bullet$ *there is a unique smooth map* $f : \Delta^d \to \Delta^n$ *such that* $m = f_\bullet$.

Proof. Define f by $m(id)$ for $id : \Delta^d \to \Delta^d$ the identity. Then

$$\begin{aligned} f_\bullet &= (m(id))_\bullet \\ &= (m(id))_* \\ &= m \circ id_* \quad \text{(as } m \text{ is smooth)} \\ &= m. \end{aligned}$$

So f induces m. Now if g induces m then $g_\bullet = m$ hence $g = g_\bullet(id) = m(id)$. □

3.1. **Smooth Kan complexes.**

Definition 3.9. *A smooth simplicial set whose underlying simplicial set is a Kan complex will be called a* **smooth Kan complex***.*

The above notion will be crucial for us.

3.2. **Some examples.** Let Y be a smooth manifold or more generally a diffeological space and let $Sing^{sm}(Y)$ denote the simplicial set of smooth singular simplices in Y [6]. That is $Sing^{sm}(Y)(k)$ is the set of smooth maps $\Sigma : \Delta^k \to Y$, with its natural simplicial structure. $Sing^{sm}(Y)$ will often be abbreviated by $Y_\bullet$.

Example 3.10. *Let* Y *be a smooth manifold, or a diffeological space. Set* $X = Y_\bullet$*, then* X *is naturally a smooth simplicial set. When* Y *is a finite dimensional manifold,* X *is a smooth Kan complex, this is essentially* [5, Corollary 4.36]*. Only "essentially", as the latter uses "non-compact simplices". We will however not need this and so will not elaborate.*

Example 3.11. *Here is one special example. Let* M *be a smooth manifold. Then there is a natural smooth simplicial set* LM^Δ *whose* d*-simplices* Σ *are smooth maps* $f_\Sigma : \Delta^d \times S^1 \to M$*. The maps* Σ_* *are defined by*

$$\Sigma_*(\sigma) = f_\Sigma \circ (\sigma \times id),$$

for $\sigma \in \Delta^d_\bullet(k)$ *and*

$$\sigma \times id : \Delta^k \times S^1 \to \Delta^d \times S^1,$$

[6]This is often called the "smooth singular simplicial set of Y". However, for us "smooth" is reserved for another purpose, so to avoid confusion we do not use such terminology.

the product map. This LM^Δ is one simplicial model of the free loop space. Naturally, the free loop space LM also has the structure of a Fréchet manifold, in particular we have the smooth simplicial set $LM_\bullet$, whose n-simplices are Gateaux C^∞ maps $\Sigma: \Delta^n \to LM$, see Hamilton [12]. There is a natural simplicial map $LM^\Delta \to LM_\bullet$, which is readily seen to be smooth. (It is indeed a diffeomorphism.)

The above smooth simplicial set structure LM^Δ, in the language of diffeologies, is closely related to the functional diffeology on $C^\infty(Y,Z)$, for which there are diffeological diffeomorphisms:

$$C^\infty(X \times Y, Z) \to C^\infty(X, C^\infty(Y,Z)),$$

given another diffeological space X.

3.3. **Smooth simplex category of a smooth simplicial set.** Given a smooth simplicial set X, there is an extension of the previously defined simplex category $\Delta(X)$.

Definition 3.12. *For X a smooth simplicial set, $\Delta^{sm}(X)$ will denote the category whose set of objects* $\operatorname{obj}\Delta^{sm}(X)$ *is the set of smooth maps*

$$\Sigma: \Delta^d_\bullet \to X, \quad d \geq 0$$

and morphisms $f: \Sigma_1 \to \Sigma_2$, commutative diagrams:

$$\begin{array}{ccc} \Delta^d_\bullet & \xrightarrow{\widetilde{f}_\bullet} & \Delta^n_\bullet \\ & \searrow^{\Sigma_1} & \downarrow{\scriptstyle \Sigma_2} \\ & & X \end{array} \tag{3.5}$$

with top arrow any smooth map (for the tautological smooth simplicial set structure on $\Delta^d_\bullet$), which we denote by $\widetilde{f}_\bullet$. By Lemma 3.8, $\widetilde{f}_\bullet$ is induced by a unique smooth map $\widetilde{f}: \Delta^d \to \Delta^n$.

By Proposition 3.7 we have a natural faithful embedding $\Delta(X) \to \Delta^{sm}(X)$ that is an isomorphism on object sets. We call elements of $\Delta^{sm}(X)$ d-simplices.

Proposition 3.13. *Definitions 3.1, 3.3 are equivalent.*

Proof. Let $\mathcal{C}_1$ denote the category of smooth simplicial sets as given by the Definition 3.1. And let $\mathcal{C}_2$ denote the category of smooth simplicial sets as given by the Definition 3.3.

Given $X \in \mathcal{C}_1$, we define a functor $I(X): (\Delta^{sm})^{op} \to Set$ by setting

$$I(X)(k) = \{\Sigma_\bullet : \Delta^k_\bullet \to X \,|\, \Sigma_\bullet \text{ is smooth i.e. is a morphism in } \mathcal{C}_1\}.$$

And for $\sigma: \Delta^k \to \Delta^d$ a smooth map setting

$$I(X)(\sigma): I(X)(d) \to I(X)(k)$$

to be the map

$$I(X)(\sigma)(\Sigma_\bullet) = \Sigma_\bullet \circ \sigma_\bullet. \tag{3.6}$$

This defines

$$I: \mathcal{C}_1 \to \mathcal{C}_2$$

on objects.

Conversely, given $F \in \mathcal{C}_2$, define a simplicial set $I^{-1}(F)$ by the rules:

(1) $I^{-1}(F)(k) := F(k)$.
(2) For $\Sigma \in I^{-1}(F)(k)$, $\Sigma_* : \Delta^k_\bullet \to X$ is the map:
$$\Sigma_*(\sigma) = F(\sigma)(\Sigma).$$

So that we get an element $I^{-1}(F) \in \mathcal{C}_1$. This defines
$$I^{-1} : \mathcal{C}_2 \to \mathcal{C}_1$$
on objects. By Proposition 3.7 $(I^{-1} \circ I(X)) \simeq X$, an isomorphism in $\mathcal{C}_1$.

Suppose now we are given a morphism in $\mathcal{C}_1$: $f : X_0 \to X_1$ i.e. a simplicial map satisfying the condition:
$$\forall n \in \mathbb{N}\, \forall \Sigma \in X(n) : (f(\Sigma))_* = f \circ \Sigma_*. \tag{3.7}$$
Define a natural transformation:
$$I(f) : I(X_0) \to I(X_1),$$
by setting $I(f)_k : I(X_0)(k) \to I(X_1)(k)$ to be the map $I(f)_k(\Sigma_\bullet) = f \circ \Sigma_\bullet$.

This is a natural transformation by the associativity of the composition $f \circ (\Sigma_\bullet \circ \sigma_\bullet) = (f \circ \Sigma_\bullet) \circ \sigma_\bullet$.

It is clear that $I : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor. We show that it is faithful on hom sets. If $f_0, f' : X_0 \to X_1$ are a pair of morphisms in $\mathcal{C}_1$ suppose that $I(f) = I(f')$. Then
$$\forall n \in \mathbb{N}\, \forall \Sigma_\bullet \in I(X)(n) : f \circ \Sigma_\bullet = f' \circ \Sigma_\bullet.$$
In particular,
$$\forall n \in \mathbb{N}\, \forall \Sigma \in X(n) : f \circ \Sigma_* = f' \circ \Sigma_*,$$
as $\Sigma_* \in I(X)(n)$. And so
$$\forall n \in \mathbb{N}\, \forall \Sigma \in (X)(n) : f(\Sigma) = f'(\Sigma).$$
And thus $f = f'$.

We show that I surjective on hom sets. Suppose that $N : I(X_0) \to I(X_1)$ is a morphism in $\mathcal{C}_2$, i.e. a natural transformation of the corresponding functors. So for $\sigma : \Delta^d \to \Delta^k$ smooth, we have a commutative diagram:
$$\begin{array}{ccc} I(X_0)(k) & \xrightarrow{I(X_0)(\sigma)} & I(X_0)(d) \\ \downarrow N_k & & \downarrow N_d \\ I(X_1)(k) & \xrightarrow{I(X_1)(\sigma)} & I(X_2)(d) \end{array} \tag{3.8}$$

Define a simplicial map
$$f_N : X_0 \to X_1,$$
by
$$f_N(\Sigma) = N_k(\Sigma_*)|_{\Delta^k_{simp}},$$
for $\Sigma \in X_0(k)$.

We check that $I(f_N) = N$. Let $\Sigma_\bullet : \Delta^d_\bullet \to X_0$ be smooth. For $\sigma : \Delta^k \to \Delta^d$ smooth, we have:

$$\begin{aligned}
I(f_N)_d(\Sigma_\bullet)(\sigma) &= (f_N \circ \Sigma_\bullet)(\sigma), \text{ by definition of } I \\
&= f_N(\Sigma_\bullet(\sigma)) \\
&= N_k(\Sigma_\bullet(\sigma)_*)|_{\Delta^k_{simp}}, \text{ by definition of } f_N \\
&= N_k(\Sigma_\bullet \circ \sigma_\bullet)|_{\Delta^k_{simp}}, \text{ as } \Sigma_\bullet \text{ is smooth} \\
&= N_d(\Sigma_\bullet) \circ \sigma_\bullet|_{\Delta^k_{simp}}, \text{ by } N \text{ being a natural transformation, (3.8) and (3.6)} \\
&= N_d(\Sigma_\bullet) \circ \sigma, \text{ notation 2.2} \\
&= N_d(\Sigma_\bullet)(\sigma), \text{ identification (2.1).}
\end{aligned}$$

Since $\Sigma_\bullet, \sigma$ were general it follows that $I(f_N) = N$.

We have proved that I is a functor that is essentially surjective on objects, and is fully-faithful on hom sets, it follows by a classical theorem of category theory that I is an equivalence of categories. □

3.4. **Products.** Given a pair of smooth simplicial sets $(X_1, g_1), (X_2, g_2)$, the product $X_1 \times X_2$ of the underlying simplicial sets, has the structure of a smooth simplicial set

$$(X_1 \times X_2, g_1 \times g_2),$$

constructed as follows. Denote by $\pi_i : X_1 \times X_2 \to X_i$ the simplicial projection maps. Then for each $\Sigma \in (X_1 \times X_2)(d)$,

$$(g_1 \times g_2)(\Sigma) : \Delta^d_\bullet \to X_1 \times X_2$$

is defined by:

$$(g_1 \times g_2)(\Sigma)(\sigma) := (g_1(\pi_1(\Sigma))(\sigma), g_2(\pi_2(\Sigma))(\sigma)).$$

3.5. **More on smooth maps.** As defined, a smooth map $f : X \to Y$ of smooth simplicial sets, induces a functor

$$\Delta^{sm} f : \Delta^{sm}(X) \to \Delta^{sm}(Y).$$

This is defined by $\Delta^{sm} f(\Sigma) = f \circ \Sigma$, where $\Sigma : \Delta^d_\bullet \to X$ is in $\Delta^{sm}(X)$. If $m : \Sigma_1 \to \Sigma_2$ is a morphism in $\Delta^{sm}(X)$:

$$\begin{array}{ccc}
\Delta^k_\bullet & \xrightarrow{\widetilde{m}_\bullet} & \Delta^d_\bullet \\
& \searrow^{\Sigma_1} & \downarrow{\Sigma_2} \\
& & X,
\end{array}$$

then obviously the diagram below also commutes:

$$\begin{array}{ccc}
\Delta^k_\bullet & \xrightarrow{\widetilde{m}_\bullet} & \Delta^d_\bullet \\
& \searrow^{h_1} & \downarrow{h_2} \\
& & Y,
\end{array}$$

where $h_i = \Delta^{sm} f(\Sigma_i) = f \circ \Sigma_i$, $i = 1, 2$. And so the latter diagram determines a morphism $\Delta^{sm} f(m) : h_1 \to h_2$ in $\Delta^{sm}(Y)$. Clearly, this determines a functor $\Delta^{sm} f$ as needed.

3.6. **Smooth homotopy.**

Definition 3.14. *Let X, Y be smooth simplicial sets. Set $I := \Delta^1_\bullet$ and let $I_0, I_1 \subset I$ be the images of the pair of inclusions $\Delta^0_\bullet \to I$ corresponding to the pair of endpoints. A pair of smooth maps $f, g : X \to Y$ are called* **smoothly homotopic** *if there exists a smooth map*

$$H : X \times I \to Y$$

such that $H|_{X \times I_0} = f$ and $H|_{X \times I_1} = g$. H will be called a smooth homotopy between f, g.

Let X be a smooth simplicial set and $x_0 \in X(0)$. We say that a smooth $f : \Delta^n_\bullet \to X$ is ***relative to*** x_0 if $f|_{\partial \Delta^n_\bullet}$ has image in $x_{0,\bullet}$, where $x_{0,\bullet}$ denotes the image of $\Delta^0_\bullet \to X$ determined by x_0, and where $\partial \Delta^n_\bullet$ is the sub-simplicial set corresponding to simplices with image in $\partial \Delta^n$. We may analogously define $F : \Delta^n_\bullet \times I \to X$ to be relative to x_0, if $\partial \Delta^n_\bullet \times I$ has image in $x_{0,\bullet}$. We call this a ***relative homotopy***. Then we have:

Definition 3.15. *Set $\pi^{sm}_k(X, x_0)$ to be the set of equivalence classes of smooth relative to x_0 maps $f : \Delta^k_\bullet \to X$, where $f \sim g$ if there is a smooth relative homotopy $H : \Delta^k_\bullet \times I \to X$, between f, g.*

Remark 3.16. *When X is a Kan complex $\pi^{sm}_k(X, x_0)$ can be shown to be a group, but we will not need this.*

3.7. **Geometric realization.**

Geometric realization of a smooth simplicial set X is defined to be the geometric realization of the underlying simplicial set.

4. Differential forms on smooth simplicial sets

The theory of differential forms on smooth simplicial sets that we now present is part of the standard abstract theory of differential forms on simplicial sets. Some of the results of this are folklore, for example the de Rham theorem can be credited to Sullivan [46], but many much more detailed, subsequent expositions have been made, for example DuPont [7]. As such, the theory of differential forms here is a priori *inequivalent* to the theory of differential forms on diffeological spaces in the sense of Souriau [45]. If one wanted to translate our discussion of differential forms into the language of diffeological spaces, then probably it would be similar to the work of Katsuhiko [21], see also [22], [15].

First we define smooth differential forms on the topological simplices Δ^d.

Definition 4.1. *Set $T\Delta^d := i^* T\mathbb{R}^d$ for $i : \Delta^d \to \mathbb{R}^d$ the natural inclusion. Let $T^*\Delta^d$ denote the dual vector bundle. A* **smooth differential k-form** *ω on Δ^d is a section of $\Lambda^k(T^*\Delta^d)$, having a smooth extension to a section of $\Lambda^k(T^*N)$ for $N \supset \Delta^d$ an open subset of $\mathbb{R}^d$.*

The above is equivalent to various other possible definitions. For example we may take Δ^d to be a special case of a smooth manifold with corners, and use a more general theory of differential forms. This can be done, for example, using theory of diffeological spaces [11]. See also Karshon-Watts [17], which establishes one kind of "uniqueness of notions of smooth structures" for the case of simplices, so that our chosen model is canonical up to suitable equivalence.

Definition 4.2. *Let X be a simplicial set. A* ***simplicial differential k-form*** *ω, or just differential k-form where there is no possibility of confusion, is an assignment for each d-simplex Σ of X a smooth differential k-form $\omega(\Sigma) = \omega_\Sigma$ on Δ^d, such that*

$$i^* \omega_{\Sigma_2} = \omega_{\Sigma_1}, \tag{4.1}$$

for every morphism $i : \Sigma_1 \to \Sigma_2$ in $\Delta(X)$, (see Section 2.2). If in addition X is a smooth simplicial set, and if in addition:

$$i^* \omega_{\Sigma_2} = \omega_{\Sigma_1}, \tag{4.2}$$

for every morphism $i : \Sigma_1 \to \Sigma_2$ in $\Delta^{sm}(X)$ then we say that ω is **coherent**.

Remark 4.3. *In the main applications here coherence will be unnecessary, and so will not be assumed. We can sharpen our constructions to get universal Chern-Weil forms that are coherent, but this will add much length to the paper, and is only interesting in more in depth applications so is postponed. (To get coherence, in the construction of the universal bundles we must include connections as part of the data, somewhat akin to what is done in Freed-Hopkins [9].)*

A simplicial differential form ω may be denoted simply as $\omega = \{\omega_\Sigma\}$. It may also be convenient to use the anonymous function notation $\Sigma \mapsto \omega_\Sigma$.

Example 4.4. *If Y is a smooth d-fold, and if ω is a differential k-form on Y, then $\Sigma \mapsto \Sigma^* \omega$ is a coherent simplicial differential k-form on $Y_\bullet$ called the* **induced simplicial differential form**. *And this determines a dg map:*

$$\Theta : \Omega^\bullet(Y, \mathbb{R}) \to \Omega^\bullet(Y_\bullet, \mathbb{R}). \tag{4.3}$$

Example 4.5. *Let LM^Δ be the smooth Kan complex of Example 3.11. Then Chen's iterated integrals [3] naturally give coherent differential forms on LM^Δ. More specifically, each d-simplex of LM^Δ corresponds to a smooth "plot" of the form $\Delta^d \to LM$ (in Chen's language). Chen's iterated integrals as differential forms on LM, amount to a rule in particular giving a differential forms on Δ^d, for each such plot. The coherence condition in our language is implied by the condition [3,* Definition 1.2.2] *for this rule. So that this exactly gives a coherent differential form in our language.*

Let X be a simplicial set. We denote by $\Omega^k(X)$ the $\mathbb{R}$-vector space of differential k-forms on X. Define

$$d : \Omega^k(X) \to \Omega^{k+1}(X)$$

so that $d(\omega)$ abbreviated by $d\omega$ is:

$$\Sigma \mapsto d\omega_\Sigma.$$

Clearly we have

$$d^2 = 0.$$

A k-form ω is said to be ***closed*** if $d\omega = 0$, and ***exact*** if for some $(k-1)$-form η, $\omega = d\eta$.

Definition 4.6. *The* **wedge product** *on*

$$\Omega^\bullet(X) = \bigoplus_{k \geq 0} \Omega^k(X)$$

is defined by

$$\omega \wedge \eta(\Sigma) = \omega_\Sigma \wedge \eta_\Sigma.$$

Then $\Omega^\bullet(X)$ has the structure of a differential graded $\mathbb{R}$-algebra with respect to $\wedge$.

We then, as usual, define the ***de Rham cohomology*** of X:

$$H^k_{DR}(X) = \frac{\text{closed k-forms}}{\text{exact k-forms}},$$

then

$$H^\bullet_{DR}(X) = \bigoplus_{k \geq 0} H^k_{DR}(X)$$

is a graded commutative $\mathbb{R}$-algebra.

Versions of the simplicial de Rham complex have been used by Whitney and perhaps most famously by Sullivan [46]. In particular, the proof of the de Rham theorem (next section) is due to Sullivan.

4.1. **Homology and cohomology of a simplicial set.** We go over this mostly to establish notation. For a simplicial set X, we define an abelian group

$$C_k(X, \mathbb{Z}),$$

as the free abelian group generated by the set of k-simplices $X(k)$. Elements of $C_k(X, \mathbb{Z})$ are called k-***chains***. The boundary operator:

$$\partial : C_k(X, \mathbb{Z}) \to C_{k-1}(X, \mathbb{Z}),$$

is defined on a k-simplex σ by

$$\partial \sigma = \sum_{i=0}^{n} (-1)^i d_i \sigma,$$

where $d_i : X(k) \to X(k-1)$ are the face maps, this is then extended by linearity to general chains. Then clearly $\partial^2 = 0$.

The homology of this complex is denoted by $H_k(X, \mathbb{Z})$, called integral homology. The integral cohomology is defined analogously to the standard topology setting, using dual chain groups $C^k(X, \mathbb{Z}) = hom(C_k(X, \mathbb{Z}), \mathbb{Z})$. The corresponding coboundary operator is denoted by d as usual:

$$d : C^k(X, \mathbb{Z}) \to C^{k+1}(X, \mathbb{Z}).$$

Homology and cohomology with other ring coefficients (or modules) are defined analogously. Given a simplicial map $f : X \to Y$ there are natural induced chain maps $f^* : C^k(Y, \mathbb{Z}) \to C^k(X, \mathbb{Z})$, and $f_* : C_k(X, \mathbb{Z}) \to C_k(X, \mathbb{Z})$.

We say that a pair of simplicial maps $f, g : X \to Y$ are ***homotopic*** if there is a simplicial map $H : X \times \Delta^1_{simp} \to Y$ so that $f = H \circ i_0$, $g = H \circ i_1$ for $i_0, i_1 : X \to X \times \Delta^1_{simp}$ corresponding to the pair of end point inclusions $\Delta^0_{simp} \to \Delta^1_{simp}$. A ***simplicial homotopy equivalence*** is then defined analogously to the topological setting.

As is well known if f, g are homotopic then f^*, g^* and f_*, g_* are chain homotopic.

4.2. **Integration.** Let X be a simplicial set. Given a chain

$$\sigma = \sum_i a_i \Sigma_i \in C_k(X, \mathbb{Z})$$

and a smooth differential form ω, we define:

$$\int_\sigma \omega = \sum_i a_i \int_{\Delta^k} \omega_{\Sigma_i}$$

where the integrals on the right are the standard integrals of differential forms. Thus, we obtain a homomorphism:

$$\int : \Omega^k(X) \to C^k(X, \mathbb{R}),$$

$\int(\omega)$ is the k-cochain defined by:

$$\int(\omega)(\sigma) := \int_\sigma \omega,$$

where σ is a k-chain. We will abbreviate $\int(\omega)$ by $\int \omega$. The following is well known.

Lemma 4.7. *For a simplicial set X, the homomorphism $\int$ commutes with d, and so induces a homomorphism:*

$$DR : \Omega^\bullet(X) \to C^\bullet(X, \mathbb{R}),$$

with the induced map on cohomology denoted by the same symbol.

Proof. We need that

$$\int d\omega = d \int \omega.$$

Let $\Sigma : \Delta^k_{simp} \to X$ be a k-simplex. Then

$$\begin{aligned} (\int d\omega)_\Sigma &= \int_{\Delta^k} d\omega_\Sigma, && \text{by definition} \\ &= \int_{\partial \Delta^k} \omega_\Sigma, && \text{by Stokes theorem} \\ &= d(\int \omega)_\Sigma, && \text{by the definition of } d \text{ on co-chains.} \end{aligned}$$

□

The de Rham theorem tells us that DR is a quasi-isomorphism see [7], but we will not need this.

4.3. **Pull-back.** Given a (smooth) map $f : X_1 \to X_2$ of (smooth) simplicial sets, we define

$$f^* : \Omega^k(X_2) \to \Omega^k(X_1)$$

naturally by

$$f^*(\omega)(\Sigma) := \omega(f(\Sigma)). \tag{4.4}$$

Let's check that f^* commutes with d. We have:

$$\begin{aligned}\forall \Sigma : f^*(d\omega)(\Sigma) &= d\omega(f(\Sigma)) \\ &= d(f^*\omega(\Sigma)) \\ &= d(f^*\omega)(\Sigma).\end{aligned}$$

So we have an induced differential graded $\mathbb{R}$-algebra homomorphism:

$$f^* : \Omega^\bullet(X_2) \to \Omega^\bullet(X_1).$$

And in particular an induced $\mathbb{R}$-algebra homomorphism:

$$f^* : H^\bullet_{DR}(X_2) \to H^\bullet_{DR}(X_1).$$

4.4. **Relation with ordinary homology and cohomology.** Let $s-Set$ denote the category of simplicial sets and Top the category of topological spaces. Let

$$|\cdot| : s-Set \to Top$$

be the geometric realization functor as defined in Section 2.3. Let X be a (smooth) simplicial set. Then for any ring K and any $d \in \mathbb{N}$ we have natural chain maps

$$\begin{aligned}&CR : C_d(X, K) \to C_d(|X|, K), \\ &CR^\vee : C^d(|X|, K) \to C^d(X, K).\end{aligned} \tag{4.5}$$

The chain map CR is defined as follows. A d-simplex $\Sigma : \Delta^d_{simp} \to X$, by construction of $|X|$ naturally induces a continuous map $\Sigma_{top} : \Delta^d \to |X|$. Denote by also by Σ_{top} the corresponding generator of $C_d(|X|, K)$, then we set $CR(\Sigma) = \Sigma_{top}$ in this notation. Then $CR^\vee$ is the dual chain map.

CR and $CR^\vee$ are quasi-isomorphisms, i.e. induce isomorphisms

$$\begin{aligned}&R : H_d(X, K) \to H_d(|X|, K), \\ &R^\vee : H^d(|X|, K) \to H^d(X, K).\end{aligned} \tag{4.6}$$

Remark 4.8. *This works as follows. Let $|X|^f$ denote the geometric realization of X omitting the degeneracies in the colimit construction, (that is we take the colimit over the category of face maps). Then $|X|^f$ is an infinite dimensional Δ-complex, and as shown by Hatcher* [13, Section 2.1], *the Δ-complex homology of a Δ-complex is isomorphic to its singular homology. On the other hand, for $|X|^f$ the Δ-complex homology is naturally identified with $H_d(X, K)$. Since $|X|^f$ is weakly equivalent to $|X|$* [42, Appendix A], *this readily implies the claim.*

Now let Y be a (finite dimensional, possibly with corners) smooth manifold and $X = Y_\bullet = Sing^{sm}(Y)$. The natural map

$$h : |Y_\bullet| \to Y \tag{4.7}$$

is a weak homotopy equivalence. To see this let $f : S^n \to Y$ represent a class in $\pi_n(Y, y_0)$, then there is a smooth $f' : S^n \to Y$ representing the same class. It readily follows that the class $[f']$ is in the image of $h_* : \pi_k(|Y_\bullet|, h^{-1}(y_0)) \to \pi_k(Y, y_0)$. Injectivity of h_* is verified similarly. So h is a homotopy equivalence by the Whitehead's theorem.

Let us denote by

$$N : Y \to |Y_\bullet|, \tag{4.8}$$

a homotopy inverse.

Define

$$I : H_d(Y_\bullet, K) \to H_d(Y, K)$$

to be the map induced by the chain map

$$CI : C_d(Y_\bullet, K) \to C_d(Y, K)$$

sending the generator of $C_d(Y_\bullet, K)$, corresponding to a simplex $\Sigma \in Y_\bullet(d)$, to the generator of $C_d(Y)$, corresponding to the smooth map $\Sigma_{top} : \Delta^d \to Y$ (as $\Sigma \in Y_\bullet(d)$ by definition uniquely corresponds to such a smooth map).

Then factor R as:

$$\begin{array}{ccc} H_d(Y_\bullet, K) & \xrightarrow{I} & H_d(Y, K) \\ & \searrow^{R} & \downarrow N_* \\ & & H_d(|Y_\bullet|, K). \end{array}$$

We may factor $R^\vee$ as:

$$\begin{array}{ccc} H^d(|Y_\bullet|, K) & \xrightarrow{N^*} & H^d(Y, K) \\ & \searrow^{R^\vee} & \downarrow I^\vee \\ & & H^d(Y_\bullet, K), \end{array} \tag{4.9}$$

where $I^\vee$ is induced by the dual $CI^\vee$ of CI.

Notation 4.9. *Let $\alpha \in H^d(X, K)$.*

(1) *We set*

$$|\alpha| := (R^\vee)^{-1}(\alpha) \in H^d(|X|, K).$$

(2) *If Y is a smooth manifold, and $X = Y_\bullet$. We set*

$$|\alpha|_{sm} := (I^\vee)^{-1}(\alpha) \in H^d(Y, K).$$

Given a map of simplicial sets $f : X_1 \to X_2$ we let $|f| : |X_1| \to |X_2|$ denote the induced map of geometric realizations.

Lemma 4.10. *Let $f : X_1 \to X_2$ be a simplicial map of simplicial sets. Let $f^* : H^d(X_2, K) \to H^d(X_1, K)$ be the induced homomorphism then:*

$$|f^*(\alpha)| = |f|^*(|\alpha|).$$

Proof. We have a clearly commutative diagram of chain maps (omitting the coefficient ring):

$$\begin{array}{ccc} C_d(X_1) & \xrightarrow{CR} & C_d(|X_1|) \\ \downarrow f_* & & \downarrow |f|_* \\ C_d(X_2) & \xrightarrow{CR} & C_d(|X_2|), \end{array}$$

from which the result immediately follows. □

The following is immediate from definitions.

Lemma 4.11. *If $K = \mathbb{R}$ then*

$$I^\vee \circ DR^{ord} = DR \circ H\Theta,$$

where:

- $DR^{ord} : H^d_{DR}(Y, \mathbb{R}) \to H^d(Y, \mathbb{R})$ *is the ordinary de Rham integration isomorphism.*
- DR *is as in Lemma 4.7.*
- $H\Theta : H^d_{DR}(Y, \mathbb{R}) \to H^d_{DR}(Y_\bullet, \mathbb{R})$ *is the cohomology map induced by the map* Θ *as in (4.3).*

5. Smooth simplicial G-bundles

In what follows G may be assumed to be either a locally convex Lie group or a diffeological Lie group, with smoothness interpreted in the corresponding categories, and where the diffeology on Δ^n is the subspace diffeology.

In Section 7 we specialize to G being a generalized Lie group, for some basics on the subject of infinite dimensional Lie groups we refer the reader to Neeb [34]. We now introduce the basic building blocks for simplicial G-bundles.

Definition 5.1. *Let P be a topological principal G-bundle over Δ^n, with the embedding $\Delta^n \subset \mathbb{R}^n$ as previously. Suppose we have a choice of a maximal atlas of topological G-bundle trivializations $\phi_i : U_i \times G \to P$, $U_i \subset \Delta^n$ open, s.t. the transitions maps*

$$(U_i \cap U_j) \times G \xrightarrow{\phi_{ij} = \phi_j^{-1} \circ \phi_i,} (U_i \cap U_j) \times G$$

extend to smooth maps $N \times G \to N \times G$, for $N \supset U_i \cap U_j$ some open set in $\mathbb{R}^n$. Then with such a choice of an atlas we call P **a smooth G-bundle over Δ^n**. *Smooth bundle maps, and isomorphisms are then defined as with standard smooth bundles.*

At this point our terminology may partially clash with common terminology, in particular a simplicial G-bundle will *not* be a presheaf on Δ with values in the category of smooth G-bundles. Instead, it will be a functor (not a co-functor!) on $\Delta^{sm}(X)$ with additional properties. Presheafs of this type will not appear in the paper so that this should not cause confusion.

In the definition of simplicial differential forms we omitted coherence. In the case of simplicial G-bundles, the analogous condition (full functoriality on $\Delta^{sm}(X)$) turns out to be necessary if we want universal simplicial G-bundles with expected behavior.

Notation 5.2. *Let G be as above, we denote by $\mathcal{G}$ the category of smooth principal G-bundles over the simplices Δ^n, (n not fixed) with morphisms smooth G-bundle maps.*

Let $\mathcal{F}_1 : \Delta^{sm}(X) \to \Delta^{sm}$ be the natural forgetful functor. And $\mathcal{F}_2 : \mathcal{G} \to \Delta^{sm}$ the functor taking a G-bundle $P \to \Delta^k$ to k and defined on morphisms as follows. If $\widetilde{\phi} : P_1 \to P_2$ is a morphism in $\mathcal{G}$ over a smooth map $\phi : \Delta^k \to \Delta^n$ then $\mathcal{F}_2(\widetilde{\phi})(k,n) = \phi$.

Definition 5.3. *Let G be as above and X a smooth simplicial set. A* **smooth simplicial G-bundle** *P over X is a functor $P : \Delta^{sm}(X) \to \mathcal{G}$, so that the diagram:*

$$\begin{array}{ccc} \Delta^{sm}(X) & \xrightarrow{P} & \mathcal{G} \\ & \searrow^{\mathcal{F}_1} & \downarrow{\scriptstyle \mathcal{F}_2} \\ & & \Delta^{sm}, \end{array}$$

commutes. We will call this the **compatibility condition**.

We will only deal with smooth simplicial G-bundles, and so will usually say ***simplicial G-bundle***, omitting the qualifier 'smooth'.

Notation 5.4. *To reduce the use of the parenthesis, we often use notation P_Σ for $P(\Sigma)$. Note that this notation is used exclusively for objects. If we write a simplicial G-bundle $P \to X$, this means that P is a simplicial G-bundle over X in the sense above. So that $P \to X$ is just notation not a morphism.*

Example 5.5. *If X is a smooth simplicial set and G is as above, we denote by $X \times G$ the simplicial G-bundle,*

$$\forall n \in \mathbb{N}, \forall \Sigma^n \in \Delta(X) : (X \times G)_{\Sigma^n} \text{ is the trivial bundle } \Delta^n \times G \to \Delta^n.$$

And where for $\sigma : \Delta^n \to \Delta^k$, $(X \times G)(\sigma) : \Delta^n \times G \to \Delta^k \times G$ is the map $\sigma \times id$.

This is called the ***trivial simplicial G-bundle over X***.

Example 5.6. *Let $Z \to Y$ be a diffeological G-bundle over a diffeological space Y. Or a smooth G-bundle over a smooth manifold Y with G locally convex. Then we have a simplicial G-bundle Z^Δ over $Y_\bullet$ defined by the conditions:*

1. $Z^\Delta_\Sigma = \Sigma^* Z$.
2. *For $f : \Sigma_1 \to \Sigma_2$ a morphism, the bundle map*
$$Z^\Delta(f) : Z^\Delta_{\Sigma_1} \to Z^\Delta_{\Sigma_2}$$
is the universal map $u : \Sigma_1^* Z \to \Sigma_2^* Z$ corresponding to the universal pullback property of $\Sigma_2^* Z$.

The uniqueness of the universal maps readily implies that Z^Δ is a functor. We say that Z^Δ is **the simplicial G-bundle induced by Z**.

Definition 5.7. *Let $P_1 \to X_1, P_2 \to X_2$ be a pair of simplicial G-bundles. Let $h : X_1 \to X_2$ be a smooth map. A* **smooth simplicial G-bundle map over h** *from P_1 to P_2 is a natural transformation of functors:*

$$\widetilde{h} : P_1 \to P_2 \circ \Delta^{sm} h,$$

such that the following additional property is satisfied. For each d-simplex $\Sigma \in \Delta^{sm}(X_1)$ the natural transformation $\widetilde{h}$ specifies a morphism in $\mathcal{G}$:

$$\widetilde{h}_\Sigma : P_1(\Sigma) \to P_2(h \circ \Sigma),$$

and we ask that this is a bundle map over the identity, so that the following diagram commutes:

$$\begin{array}{ccc} P_1(\Sigma) & \xrightarrow{\widetilde{h}_\Sigma} & P_2(h \circ \Sigma) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ \Delta^d & \xrightarrow{id} & \Delta^d . \end{array}$$

We will usually say simplicial G-bundle map instead of smooth simplicial G-bundle map, (as everything is always smooth) when h is not specified it is assumed to be the identity.

Definition 5.8. *Let P_1, P_2 be simplicial G-bundles over X_1, X_2 respectively. A* **simplicial G-bundle isomorphism** *is a simplicial G-bundle map*

$$\widetilde{h} : P_1 \to P_2$$

s.t. there is a simplicial G-bundle map

$$\widetilde{h}^{-1} : P_2 \to P_1$$

with

$$\widetilde{h}^{-1} \circ \widetilde{h} = id.$$

Usually $X_1 = X_2$ and in this case, unless specified otherwise, it is assumed $h = id$. A simplicial G-bundle isomorphic to the trivial simplicial G-bundle is called ***trivializeable***.

Definition 5.9. *If $X = Y_\bullet$ for Y a smooth manifold, we say that a simplicial G-bundle P over X is* **inducible by a smooth G-bundle** $N \to Y$ *if there is a simplicial G-bundle isomorphism $N^\Delta \to P$.*

The following will be one of the crucial ingredients later on. (Recall also that we are treating two distinct cases simultaneously: a diffeological G and locally convex G, the proof of the following theorem works the same way in both cases.)

Theorem 5.10. *Let G be as above and let $P \to Y_\bullet$ be a simplicial G-bundle, for Y a smooth d-manifold (or a manifold with corners, understood as previously). Then P is inducible by some smooth G-bundle $N \to Y$.*

Proof. We need to introduce an auxiliary notion. Let Z be a smooth d-manifold with corners, as before understood as a diffeological space. And let $\mathcal{D}(Z)$ denote the category whose objects are smooth (diffeological) embeddings $\Sigma : \Delta^d \to Z$, (for the same fixed d). A morphism $f \in hom_{\mathcal{D}(Z)}(\Sigma_1, \Sigma_2)$ is a commutative diagrams:

$$\begin{array}{ccc} \Delta^d & \xrightarrow{\widetilde{f}} & \Delta^d \\ & \searrow^{\Sigma_1} & \downarrow \Sigma_2 \\ & & Z. \end{array} \tag{5.1}$$

Note that the map $\widetilde{f}$ is unique, when such a diagram exists, as Σ_i are embeddings. Thus $hom_{\mathcal{D}(Z)}(\Sigma_1, \Sigma_2)$ is either empty or consists of a single element.

Although, we state the result for manifolds with corners, for simplicity we assume here that Y is a manifold. Let $\{O_i\}_{i \in I}$ be a locally finite open cover of Y, closed under intersections, with each O_i diffeomorphic to an open ball in $\mathbb{R}^d$. Such a cover is often called a good cover of a manifold. Existence of such a cover is a folklore theorem, but a proof can be found in [8, Prop A1].

Let $\mathcal{O}$ denote the category with the set of objects $\{O_i\}$ and with morphisms inclusions. And set $C_i = \mathcal{D}(O_i)$, then clearly C_i is a full subcategory of $\Delta^{sm}(Y_\bullet)$. For each i, we have the functor

$$F_i = P|_{C_i} : C_i \to \mathcal{G}.$$

By assumption that each O_i is diffeomorphic to an open ball, O_i has an exhaustion by embedded d-simplices. This means that there is a sequence of smooth embeddings $\Sigma_j : \Delta^d \to O_i$ satisfying:

- interior image$(\Sigma_{j+1}) \supset$ image(Σ_j) for each $j \in \mathbb{N}$.
- $\bigcup_j$ image$(\Sigma_j) = O_i$.

As each element of C_i is contained in some Σ_j,

$$\Sigma_0 \to \ldots \to \Sigma_j \to \Sigma_{j+1} \to \ldots$$

forms a final sub-category of C_i. Thus, for each i, the colimit in $\mathcal{G}$:

$$P_i := \operatorname{colim}_{C_i} F_i \tag{5.2}$$

is the colimit of the sequence

$$P(\Sigma_0) \to \ldots \to P(\Sigma_j) \to P(\Sigma_{j+1}) \ldots$$

And this colimit is naturally a topological G-bundle over $O_i = \cup_i \operatorname{image}(\Sigma_i)$.

We may give P_i the structure of a smooth G-bundle, with G-bundle charts defined as follows. For each $\Sigma \in C_i$, pick a smooth trivialization:

$$\xi_\Sigma : (\Delta^d)^\circ \times G \to (P_\Sigma^\circ := P_\Sigma|_{(\Delta^d)^\circ}).$$

Then set $\phi_{\Sigma,i}$ to be the composition map

$$(\Delta^d)^\circ \times G \xrightarrow{\xi_\Sigma} P_\Sigma^\circ \xrightarrow{c_\Sigma} P_i,$$

where $c_\Sigma : (P_\Sigma = F_i(\Sigma)) \to P_i$ is the natural map in the colimit diagram of (5.2).

Lemma 5.11. *The collection $\{\phi_{\Sigma,i}\}$ forms a smooth G-bundle atlas for P_i.*

Proof. Suppose that $z \in (\operatorname{image}\phi_{\Sigma,i}) \cap (\operatorname{image}\phi_{\Sigma',i})$. Then there is a morphism $\Sigma'' \xrightarrow{m} \Sigma$ and a morphism $\Sigma'' \xrightarrow{m'} \Sigma'$ such that $z \in \operatorname{image}\phi_{\Sigma'',i}$. And such that the following composition maps coincide:

$$P^{\circ}_{\Sigma''} \xrightarrow{P(m)} P^{\circ}_{\Sigma} \xrightarrow{c_\Sigma} P_i$$

$$P^{\circ}_{\Sigma''} \xrightarrow{P(m')} P^{\circ}_{\Sigma'} \xrightarrow{c_{\Sigma'}} P_i.$$

Hence, $c^{-1}_{\Sigma'} \circ c_\Sigma = P(m') \circ P(m)^{-1}$ where the inverses are defined on the suitable open sub-domains. Thus $c^{-1}_{\Sigma'} \circ c_\Sigma$ is smooth, which clearly implies our claim.

□

So we obtain a functor

$$D : \mathcal{O} \to \mathcal{G},$$

defined by

$$D(O_i) = P_i,$$

and defined naturally on morphisms. Specifically, a morphism $O_{i_1} \to O_{i_2}$ induces a functor $C_{i_1} \to C_{i_2}$ and hence a smooth G-bundle map $P_{i_1} \to P_{i_2}$, by the naturality of the colimit.

Let $t : \mathcal{O} \to Top$ denote the tautological functor, sending the subspace O to the corresponding topological space, so that $Y = \operatorname{colim}_{\mathcal{O}} t$, where for simplicity we write equality for natural isomorphisms here and further on in this proof.

Now,

$$N := \operatorname{colim}_{\mathcal{O}} D, \tag{5.3}$$

is naturally a topological G-bundle

$$N \xrightarrow{p} \operatorname{colim}_{\mathcal{O}} t = Y.$$

Let $c_i : P_i \to N$ denote the natural maps in the colimit diagram of (5.3). The collection of charts $\{c_i \circ \phi_{\Sigma,i}\}_{i,\Sigma\in C_i}$ forms a smooth G-bundle atlas on N (repeat the argument of Lemma 5.11). Let us rename these charts as $\{\rho_k\}$, for k elements of the index set implicit above.

We now prove that P is induced by N. Let $\Sigma : \Delta^n \to Y$ be smooth, then $\{V_i := \Sigma^{-1}(O_i)\}_{i\in I}$ is a locally finite and hence finite open cover of Δ^n closed under intersections. Let N^Δ be the simplicial G-bundle over $Y_\bullet$ induced by N. So

$$N^\Delta_\Sigma := \Sigma^* N.$$

As Δ^n is a convex subset of $\mathbb{R}^n$, the open metric balls in Δ^n, for the induced metric, are convex as subsets of $\mathbb{R}^n$. Consequently, as each $V_i \subset \Delta^n$ is open, it has a basis of convex metric balls, with respect to the induced metric. By Rudin [39] there is then a locally finite cover of V_i by elements of this basis. In fact, Rudin shows any open cover of V_i has a locally finite refinement by elements of such a basis.

Let $\{W^i_j\}$ consist of elements of this cover and all intersections of its elements, (which must then be finite intersections). So $W^i_j \subset V_i$ are open convex subsets and $\{W^i_j\}$ is a locally finite open cover of V_i, closed under finite intersections.

As each $W^i_j \subset \Delta^n$ is open and convex it has an exhaustion by nested images of embedded simplices. That is

$$W^i_j = \bigcup_{k\in\mathbb{N}} \operatorname{image}\sigma^{i,j}_k \tag{5.4}$$

for $\sigma^{i,j}_k : \Delta^d \to W^i_j$ smooth and embedded, with $\operatorname{image}\sigma^{i,j}_k \subset \operatorname{image}\sigma^{i,j}_{k+1}$ for each k.

Let C be the small category with objects $I \times J \times \mathbb{N}$, so that there is exactly one morphism from $a = (i, j, k)$ to $b = (i', j', k')$ whenever $\operatorname{image}\sigma^{i,j}_k \subset \operatorname{image}\sigma^{i',j'}_{k'}$, and no morphisms otherwise. Let

$$F : C \to \mathcal{D}(\Delta^d)$$

be the functor $F(a) = \sigma^{i,j}_k$ for $a = (i, j, k)$, (the definition on morphisms is forced). For brevity, we denote $\sigma_a := F(a)$.

For a smooth manifold with corners Y, if $\mathcal{O}(Y)$ denotes the category of topological subspaces of Y with morphisms inclusions, then there is a forgetful functor

$$T : \mathcal{D}(Y) \to \mathcal{O}(Y)$$

which takes f to $\operatorname{image}(\widetilde{f})$. With all this in place, we have:

Lemma 5.12.

$$\Delta^d = \operatorname{colim}_C T \circ F, \tag{5.5}$$

as a colimit in Top.

Proof. First recall that a general topological space X is the colimit of any open cover $\{O_i\}$ of X closed under intersections. In particular, Δ^d is the colimit of the cover $\{W^i_j\}_{i,j}$. On the other hand $W^i_j = \operatorname{colim}_{S_{i,j}} T \circ F$, for $S_{i,j} \subset C$ a full subcategory corresponding to the exhaustion 5.4. Moreover, $C = \cup_{i,j} S_{i,j}$. The result readily follows. □

It follows that

$$N^\Delta_\Sigma = \operatorname{colim}_C N^\Delta \circ \Delta^{sm}\Sigma \circ F.$$

Now, by construction for each $a \in C$, $\Sigma \circ \sigma_a$ is contained in an open set O_i diffeomorphic to the standard open ball in $\mathbb{R}^d$. It follows that we may express:

$$\Sigma \circ \sigma_a = \Sigma_a \circ m_a \circ \sigma_a, \tag{5.6}$$

for some $\Sigma_a : \Delta^d \to O_i \subset Y$ a smooth embedded d-simplex. And $m_a : \Delta^n \to \Delta^d$ smooth.

So for all $a \in C$,

$$N^\Delta \circ \Delta^{sm}\Sigma \circ F(a) = (m_a \circ \sigma_a)^* P_{\Sigma_a},$$

after naturally identifying P_{Σ_a} with $N^\Delta_{\Sigma_a}$. More precisely, there is a natural isomorphism $\phi_a : P_{\Sigma_a} \to N^\Delta_{\Sigma_a}$ given by the composition:

$$P_{\Sigma_a} \to P_i \to N, \tag{5.7}$$

with the first map the bundle map in the colimit diagram of (5.2), and the second map the bundle map in the colimit diagram of (5.3). The composition (5.7) gives a bundle map over Σ_a. And so, by the defining universal property of the pull-back, there is a uniquely induced universal map

$$P_{\Sigma_a} \to (\Sigma_a)^* N = N^{\Delta}_{\Sigma_a},$$

which is a G-bundle isomorphism.

Also,

$$P_\Sigma = \mathrm{colim}_C\, P \circ \Delta^{sm}\Sigma \circ F.$$

Similarly to the above discussion we have that for all $a \in C$:

$$P \circ \Delta^{sm}\Sigma \circ F(a) = P_{\Sigma_a \circ m_a \circ \sigma_a},$$

and by functoriality of P there is a morphism:

$$P(m_a \circ \sigma_a) : P_{\Sigma_a \circ m_a \circ \sigma_a} \to P_{\Sigma_a},$$

over $m_a \circ \sigma_a$ and hence an induced natural morphism:

$$P_{\Sigma_a \circ m_a \circ \sigma_a} \to (m_a \circ \sigma_a)^* P_{\Sigma_a},$$

which is also a G-bundle isomorphism.

To summarize, we obtain for all $a \in C$ a natural isomorphism

$$N \circ \Delta^{sm}\Sigma \circ F(a) \xrightarrow{\phi_a} P \circ \Delta^{sm}\Sigma \circ F(a).$$

These fit into a natural transformation of functors:

$$\phi : N \circ \Delta^{sm}\Sigma \circ F \to P \circ \Delta^{sm}\Sigma \circ F.$$

So that ϕ induces a map of the colimits:

$$h_\Sigma : P_\Sigma \to N^{\Delta}_{\Sigma},$$

by naturality, and this is an isomorphism of these smooth G-bundles. It is then clear that $\{h_\Sigma\}_\Sigma$ determines the bundle isomorphism $h : P \to N^\Delta$ we are looking for. □

5.1. **Pullbacks of simplicial bundles.** Let $P \to X$ be a simplicial G-bundle over a smooth simplicial set X. And let $f : Y \to X$ be a smooth map of smooth simplicial sets. We define the pull-back simplicial G-bundle $f^*P \to Y$ to be the functor $f^*P := P \circ \Delta^{sm} f$.

Note that the analogue of the following lemma is not true in the category of topological fibrations. The pull-back by the composition is not the composition of pullbacks (except up to a natural isomorphism).

Lemma 5.13. *The pull-back is functorial. So that if $f : X \to Y$ and $g : Y \to Z$ are smooth maps of smooth simplicial sets, and $P \to Z$ is a smooth simplicial G-bundle over Z then*

$$(g \circ f)^* P = f^*(g^*(P)) \text{ this is an actual equality.}$$

Proof. This is of course elementary, as functor composition is associative:

$$(g \circ f)^* P = P \circ \Delta^{sm}(g \circ f) = P \circ (\Delta^{sm} g \circ \Delta^{sm} f) = (P \circ \Delta^{sm} g) \circ \Delta^{sm} f = f^*(g^* P).$$

□

6. Connections on simplicial G-bundles

In this section G is a generalized Lie group. A G-connection on a smooth G-bundle P over a finite dimensional smooth manifold X is an Ehresmann G-connection, that is a smooth, right G-invariant horizontal distribution. Existence of such connections is proved as in the case of finite dimensional bundles. One notes that in trivializations connections form an affine space, and then uses partitions of unity over the base, [7] see for instance [6].

In our setting, we only need to treat the case of G-bundles P over a simplex Δ^n. As such a G-bundle is trivializable, the space of G-connections on P is in correspondence with the space of Lie algebra valued 1-forms. The regularity condition also ensures that there is a good theory of parallel transport, see Section 7.1.1. Thus, we can completely avoid the generalities.

Definition 6.1. *A* **simplicial** G**-connection** D *on a simplicial* G*-bundle* P *over a smooth simplicial set* X *is for each* d*-simplex* Σ *of* X*, a smooth* G*-invariant Ehresmann* G*-connection* $D(\Sigma) = D_\Sigma$ *on* P_Σ*. This data is required to satisfy: if* $f : \Sigma_1 \to \Sigma_2$ *is a morphism in* $\Delta(X)$ *then*

$$P(f)^* D_{\Sigma_2} = D_{\Sigma_1}. \tag{6.1}$$

We say that D *is* **coherent** *if the same holds for all morphisms* $f : \Sigma_1 \to \Sigma_2$ *in* $\Delta^{sm}(X)$*. We will often say* G*-connection instead of simplicial* G*-connection, where there is no need to disambiguate.*

As with differential forms the coherence condition is very restrictive, and is not part of the basic definition.

Let $P \to X$ be a simplicial G-bundle. Define $P \times I \to X \times I$, for $I := [0,1]_\bullet$, to be the simplicial G-bundle $pr^* P$, for $pr : X \times I \to X$ the natural projection.

Lemma 6.2. *G-connections on simplicial G-bundles exist and any pair of G-connections D_1, D_2 on a simplicial G-bundle P are* **concordant***. The latter means that there is a G-connection on $\widetilde{D}$ on $P \times I \to X \times I$, which restricts to D_1, D_2 on $P \times I_0$, respectively on $P \times I_1$, for $I_0, I_1 \subset I$ denoting the images of the two end point inclusions $\Delta^0_\bullet \to I$.*

Proof. Suppose that $\Sigma : \Delta^d_{simp} \to X$ is a degeneracy of a 0-simplex $\Sigma_0 : \Delta^0_{simp} \to X$, meaning that there is a morphism from Σ to Σ_0 in $\Delta(X)$. Then $P_\Sigma = \Delta^d \times P_{\Sigma_0}$ (as previously equality indicates natural isomorphism) and we fix the corresponding trivial connection D_Σ on P_Σ. This assignment satisfies the condition that for all morphisms $m : \Sigma_1 \to \Sigma_2$ in $\Delta(X)$, for Σ_1, Σ_2 degeneracies of 0-simplices, $D_{\Sigma_1} = P(m)^* D_{\Sigma_2}$. We then proceed inductively.

Suppose we have constructed connections D_Σ for all k-simplices, $0 \le k \le n$, and all their degeneracies, satisfying the condition $S(n)$, which is as follows. For all morphisms $m : \Sigma_1 \to \Sigma_2$ in $\Delta(X)$, for Σ_1, Σ_2 k-simplices or their degeneracies with $0 \le k \le n$, $D_{\Sigma_1} = P(m)^* D_{\Sigma_2}$. We construct an extension D_Σ for all $(n+1)$-simplices and their degeneracies, so that this extension satisfies $S(n+1)$.

[7] This is where we used that X is finite dimensional, this can likely be relaxed.

If Σ is a non-degenerate $(n+1)$-simplex then D_Σ is already determined over the boundary of Δ^{n+1} by the defining condition (6.1). For by the hypothesis, D_Σ is already defined on all n-simplices. Then extend D_Σ over all of Δ^{n+1} arbitrarily (since the corresponding bundle is trivializable this amounts to choosing a smooth extension of a Lie algebra valued 1-form). Given a non-identity morphism of non-degenerate k-simplices $m : \Sigma_0 \to \Sigma$, $0 \leq k \leq n+1$, degree $\Sigma_0 < n+1$ and hence m maps to the boundary of Σ, i.e. to a subsimplex of degree n or less and hence by the inductive hypothesis we have that $P(m)^* D_\Sigma = D_\Sigma$.

Thus, we have extended D_Σ to all $(n+1)$-simplices, as such a simplex is either non-degenerate or is a degeneracy of an n-simplex, and in the latter case D_Σ is defined by the hypothesis.

Now, suppose we have a degeneracy $mor : \Sigma^m \to \Sigma^k$, $k < m$, $k \leq n+1$ (Σ^k can itself be degenerate). Then we have bundle map:

$$P(mor) : P(\Sigma^m) \to P(\Sigma^k).$$

And we define $D_{\Sigma^m} = P(mor)^* D_{\Sigma^k}$. The property $S(n)$ ensures that this is well defined. And so we have constructed an assignment D_Σ for all degeneracies of $(n+1)$-simplices. By construction this satisfies $S(n+1)$. And so we have completed the inductive step.

The second part of the lemma follows by an analogous argument, since we may extend D_1, D_2 to a concordance connection $\widetilde{D}$, using the above inductive procedure.

$\square$

Example 6.3. *Given a smooth G-connection D on a smooth principal G-bundle $Z \to Y$, we naturally get a simplicial G-connection on the induced simplicial G-bundle Z^Δ. Concretely, this is defined by setting D_Σ on $Z^\Delta_\Sigma = \Sigma^* Z$ to be $\widetilde{\Sigma}^* D$, for $\widetilde{\Sigma} : \Sigma^* Z \to Z$ the natural map (in the pull-back diagram). The pull-back $\widetilde{\Sigma}^* D$, is the pre-image by $\widetilde{\Sigma}$ of the corresponding distribution. This is called the* **induced simplicial connection***, and it will be denoted by D^Δ. Going in the other direction is always possible if the given simplicial G-connection in addition satisfies coherence, but we will not elaborate.*

7. Chern-Weil homomorphism

7.1. **The classical case.** To establish notation we first discuss the standard Chern-Weil homomorphism. In this section again G will be a generalized Lie group. Let $\mathfrak{g}$ denote its Lie algebra. Let P be a G-bundle over a smooth finite dimensional manifold Y. Fix a G-connection D on P.

7.1.1. *Curvature 2-form.* Associated to D we have the curvature 2-form R^D on Y, understood as a 2-form valued in the vector bundle $\mathcal{P} \to Y$, whose fiber over $y \in Y$ is $\mathcal{R}(P_y)$ - the Lie algebra of right G-invariant vector fields on P_y. In the infinite dimensional setting the definition of this 2-form is more subtle.

Remark 7.1. *No general reference is known to me. But it might be possible to adapt the finite dimensional approach of Kobayashi-Nomizu [19], [20], to the locally convex infinite dimensional setting. The potential difficulty may be in differential* *geometric details like the Bianchi identity in infinite dimensions. The approach below relies on regularity, but on the other hand it is intuitive, and there is no differential geometry just calculus. It is also, at least implicitly, the approach one takes in symplectic geometry for curvature of Hamiltonian fibrations, see* [30, Section 6.4].

Suppose first we have a trivial bundle $(U \subset \mathbb{R}^d) \times G$, with $0 \in U$, U open. Define a smooth $\mathfrak{g}$ valued one form α^D on U by:

$$\alpha^D(v) = pr_G(\widetilde{v}),$$

where $\widetilde{v}$ is the D-horizontal lift of v, $pr_G : (U \subset \mathbb{R}^d) \times G \to G$ is the projection, and where we identify $\mathfrak{g}$ with the space of right invariant vector fields.

Then for any smooth path $\gamma : [0,1] \to U$, we get a smooth path in $\mathfrak{g}$,

$$t \mapsto \xi_t = \alpha^D(\gamma'(t)).$$

We use the defining property of the regularity of G to find the unique smooth solution curve $\widetilde{\gamma} : [0,1] \to G$ satisfying:

$$\widetilde{\gamma}'(t) = \xi_t(\gamma_t).$$

Set $\phi^D(\gamma) = \widetilde{\gamma}(1)$.

This determines a map, called the holonomy map, on the smooth based loop space:

$$Hol^D : \Omega_p U \to G,$$
$$Hol^D(\gamma) = \phi^D(\gamma).$$

The regularity of G gives that Hol^D is smooth, taking the standard Fréchet manifold structure on $\Omega_p U$.

Now, for $v, w \in T_0 U$ and $h, k \in [0,1)$ let $\gamma_{hv,hw} \in \Omega_0 U$, parametrize the oriented boundary of the parallelogram, three of whose vertices are $0, hv, hw$, where the orientation on the parallelogram is $\{v, w\}$. (We need to perturb the natural piece-wise linear parametrization to be smooth, with the same image, basically making it constant near corners of the parallelogram.)

Then the mapping $(h,k) \mapsto Hol^D(\gamma_{hv,hw}) \in G$ is smooth (it is well defined with respect to any choice of the smoothing mentioned above), and of course $(0,0) \mapsto e$ (the unit). And so we may set:

$$\forall v, w \in T_0 U : R_0^D(v,w) = \frac{\partial}{\partial h \partial k}|_{(0,0)} Hol^D(\gamma_{hv,hw}) \in \mathfrak{g}, \tag{7.1}$$

where $\mathfrak{g}$ is now understood as $T_e G$. Define R_p^D analogously at other points of $p \in U$.

Given a G-bundle map $f : T_0 \to T_1$ of G-bundles over a point let

$$\operatorname{lie} f : \mathcal{R}(T_0) \to \mathcal{R}(T_1)$$

denote the induced mapping of the Lie algebras. Note that for $T_0 = T_1 = G$ any such map f uniquely corresponds to left multiplication by some $g \in G$.

Lemma 7.2. *Let $\widetilde{f}: U \times G \to U \times G$ be a smooth G-bundle map over $f: U \to U$. Let $\widetilde{f}_p$ denote the restriction of $\widetilde{f}$ to the map of the fiber over p to the fiber over $f(p)$. And let $D' = \widetilde{f}^* D$ denote the pull-back connection, then for $v, w \in T_pU$:*

$$(f^* R^D)_p = \mathrm{lie}\, \widetilde{f}_p \circ R_p^{D'} \tag{7.2}$$

$$= Ad_{g_p}(R_p^{D'}), \tag{7.3}$$

where $g_p \in G$ corresponds to $\widetilde{f}_p$ as above, and where Ad_g denotes the adjoint action by g [8].

Proof. For simplicity suppose that $p = 0$ and $f(0) = 0$. By definition of the pull-back connection we have:

$$Hol^{D'}(\gamma_{hv,hw}) = \widetilde{f}_p^{-1} \circ Hol^D(f \circ \gamma_{hv,kw}).$$

So

$$\begin{aligned} R_0^{D'}(v,w) &= \frac{\partial}{\partial h \partial k}|_{(0,0)} Hol^{D'}(\gamma_{hv,kw}) \\ &= \mathrm{lie}\, \widetilde{f}_p^{-1}(\frac{\partial}{\partial h \partial k}|_{(0,0)} Hol^D(f \circ \gamma_{hv,kw}) \\ &= \mathrm{lie}\, \widetilde{f}_p^{-1}(\frac{\partial}{\partial h \partial k}|_{(0,0)} Hol^D(\gamma_{hf_*v,kf_*w}) \\ &= \mathrm{lie}\, \widetilde{f}_p^{-1}(R_0^D(f_*v, f_*w)) \\ &= \mathrm{lie}\, \widetilde{f}_p^{-1}(f^* R_0^D(v,w)). \end{aligned}$$

Here the third equality from the top is obtained as follows. We have the composition:

$$(V \subset \mathbb{R} \times \mathbb{R}) \to \Omega_0 U \xrightarrow{\Omega f} \Omega_0 U \xrightarrow{Hol^D} G,$$

where $V \ni (0,0)$ is open, $\Omega f : \Omega_0 U \to \Omega_0 U$ is just the map $\gamma \mapsto f \circ \gamma$, and the first map is the map $(h,k) \mapsto \gamma_{hv,kw}$. Note that the differential $D(\Omega f)_0 : \Omega_0 T_0 U \to \Omega_0 T_0 U$, at the constant loop at zero (just denoted as 0), is the map $\eta \mapsto Df_0 \circ \eta$, i.e. it is the map ΩDf_0. Then apply chain rule to this composition.

□

The curvature 2-form R^D on Y is then defined as follows. Let $\widetilde{f}: U \times G \to P$, be a smooth G-bundle parametrization map over a parametrization $f: U \to Y$. Then define R^D by the condition:

$$\forall \widetilde{f}: \ (f^* R^D)_p = \mathrm{lie}\, \widetilde{f}_p \circ R_p^{\widetilde{f}^* D}.$$

It is an elementary verification using the lemma above that R^D is well defined.

[8] As it is natural here to work with right invariant vector fields, we define the adjoint action on the Lie algebra as the left G action on right invariant vector fields.

7.1.2. *The algebra $\mathcal{I}(G)$ and the homomorphism for smooth G-bundles.* We denote by $\mathcal{I}^d(G)$ the space of continuous, symmetric multilinear functionals

$$\prod_{i=1}^{i=d} \mathfrak{g} \to \mathbb{R},$$

we will just call d-tensors, fixed by the adjoint G action. Meaning that for $\rho \in \mathcal{I}^k(G)$:

$$\rho(Ad_g(\xi_1), \ldots, Ad_g(\xi_k)) = \rho(\xi_1, \ldots, \xi_k), \quad \forall g \in G, \, \xi_i \in \mathfrak{g}.$$

And set

$$\mathcal{I}(G) = \bigoplus_{d \geq 0} \mathcal{I}^d(G).$$

This forms an algebra under the symmetric product, see for instance [20, Chapter 12].

As mentioned in the introduction, in the infinite dimensional setting $\mathcal{I}(G)$ may not be freely generated, and is possibly very intricate algebraically.

Now, let $\rho \in \mathcal{I}^k(G)$. As ρ is Ad invariant, it uniquely determines a multilinear map with the same name:

$$\rho : \prod_{i=1}^{i=k} \mathcal{R}(P_y) \to \mathbb{R},$$

by taking any G-bundle map over a point $P_y \to G$ and pulling back the functionals. We may now define a closed $\mathbb{R}$-valued $2k$-form $\omega^{\rho,D}$ on Y:

(7.4)
$$\omega^{\rho,D}(v_1, \ldots, v_{2k}) = \frac{1}{2k!} \sum_{\eta \in S_{2k}} \operatorname{sign} \eta \cdot \rho(R^D(v_{\eta(1)}, v_{\eta(2)}), \ldots, R^D(v_{\eta(2k-1)}, v_{\eta_{2k}})),$$

for S_{2k} the permutation group of a set with $2k$ elements, and where $v_1, \ldots, v_{2k} \in T_y Y$.

Set

$$cw^{P,D}(\rho) = \omega^{\rho,D}.$$

In this way we get a dg map:

$$cw^{P,D} : \mathcal{I}(G) \to \Omega^\bullet(Y, \mathbb{R}).$$

Set

$$\alpha^{\rho,D} := \int \omega^{\rho,D} \in C^{2k}(Y, \mathbb{R}).$$

Then we define the Chern-Weil characteristic class:

$$c^\rho(P) := [\alpha^{\rho,D}] \in H^{2k}(Y, \mathbb{R}). \tag{7.5}$$

7.2. **Chern-Weil homomorphism for simplicial G-bundles.** Now let P be a simplicial G-bundle over a smooth simplicial set X. Fix a simplicial G-connection D on P.

For each simplex Σ^d, we have the curvature 2-form R^D_Σ of the connection D_Σ on P_Σ, defined as in the section just above. For concreteness:

$$\forall v, w \in T_z \Delta^d : R^D_\Sigma(v, w) \in \mathcal{R}(P_y),$$

for P_z the fiber of P_Σ over $z \in \Delta^d$.

As in the previous section, let $\rho \in \mathcal{I}^k(G)$. We may now define a closed, $\mathbb{R}$-valued, simplicial differential $2k$-form $\omega^{\rho,D}$ on X:

$$\omega^{\rho,D}_\Sigma(v_1, \ldots, v_{2k}) = \frac{1}{2k!} \sum_{\eta \in S_{2k}} \operatorname{sign} \eta \cdot \rho(R^D_\Sigma(v_{\eta(1)}, v_{\eta(2)}), \ldots, R^D_\Sigma(v_{\eta(2k-1)}, v_{\eta_{2k}})).$$

Set $cw^{P,D}(\rho) = \omega^{\rho,D}$. This defines a dg map:

$$cw^{P,D} : \mathcal{I}(G) \to \Omega^\bullet(X, \mathbb{R}).$$

Definition 7.3. *Let*

$$f_i : A \to \Omega^\bullet(X, \mathbb{R}),\ i = 0, 1$$

*be dg maps of differential graded $\mathbb{R}$ algebras where X is a simplicial set (we work over $\mathbb{R}$ for simplicity). We say that f_i are **geometrically homotopic** if there is a dg map:*

$$\widetilde{f} : A \to \Omega^\bullet(X \times I, \mathbb{R}),$$

satisfying $e_0 \circ \widetilde{f} = f_0$, and $e_1 \circ \widetilde{f} = f_1$, where $e_i : \Omega^\bullet(X \times I, \mathbb{R}) \to \Omega^\bullet(X)$ are the dg maps induced by the end point inclusions $pt \to \Delta^1$.

It is not hard to see that being geometrically homotopic is an equivalence relation, but we will not need this.

Lemma 7.4. *For $P \to X$ as above*

$$cw^{P,D_0} \simeq cw^{P,D_1},$$

for any pair of simplicial G-connections D_0, D_1 on P, where $\simeq$ is the geometric homotopy relation. The latter homotopy can be made to depend solely on the choice of a concordance connection $\widetilde{D}$.

Proof. For D_0, D_1 as in the statement, fix a concordance simplicial G-connection $\widetilde{D}$, between D_0, D_1, on the simplicial G-bundle $P \times I \to X \times I$, as in Lemma 6.2.

We have a diagram of dg maps:

$$\mathcal{I}(G) \xrightarrow{cw^{P,\widetilde{D}}} \Omega^\bullet(X \times I, \mathbb{R}) \xrightarrow{r_i} \Omega^\bullet(X, \mathbb{R}),$$

where

- r_i are the restriction maps corresponding to the pair of natural inclusions $X \to X \times I$.
- $r_i \circ cw^{P,\widetilde{D}} = cw^{P,D_i}$, $i = 0, 1$.

□

Definition 7.5. *Let*

$$f_i : A \to B, \, i = 0, 1$$

be dg maps of differential graded $\mathbb{R}$ *algebras* A, B. *We say that* f_i *are* A_∞ **homotopic** *if there is an* A_∞ *map:*

$$\widetilde{f} : A \to B \otimes \Omega^\bullet(I, \mathbb{R}),$$

satisfying $\widetilde{e}_0 \circ \widetilde{f} = f_0$, *and* $\widetilde{e}_1 \circ \widetilde{f} = f_1$, *where* $\widetilde{e}_i : B \otimes \Omega^\bullet(I) \to B$ *are the natural dg maps induced by the end point inclusions* $pt \to I$, *formally defined in the proof of Proposition A.1.*

A geometric homotopy induces an A_∞ homotopy of dg maps. We relegate this to the Appendix A.

Set

$$\alpha^{\rho, D} := \int \omega^{\rho, D} \in C^{2k}(X, \mathbb{R}).$$

Then in particular, the cohomology class:

$$c^\rho(P) := [\alpha^{\rho, D}] \in H^{2k}(X, \mathbb{R}),$$

is well defined, and this is called the Chern-Weil characteristic class.

Notation 7.6. *Let us denote by* cw^P *any representative of the homotopy class* $[cw^{P,D}]$. *(For smooth or simplicial* G*-bundles* P*.)*

We have the expected naturality:

Lemma 7.7. *Let* P *be a simplicial* G*-bundle over* Y, ρ *as above and* $f : X \to Y$ *a smooth simplicial map. Then*

$$f^* \circ cw^P \simeq cw^{f^* P},$$

where $\simeq$ *as before means homotopic.*

Proof. Let D be a simplicial G-connection on P. Define the pull-back connection $f^* D$ on $f^* P$ by $f^* D(\Sigma) = D_{f(\Sigma)}$. Then $f^* D$ is a simplicial G-connection on $f^* P$. Now,

$$\begin{aligned} \forall \Sigma : \omega^{\rho, f^* D}(\Sigma) &= \omega^{\rho, D}(f(\Sigma)), \text{ by definition of } f^* D \\ &= f^* \omega^{\rho, D}(\Sigma), \text{ definition (4.4)}. \end{aligned}$$

And consequently, $\omega^{\rho, f^* D} = f^* \omega^{\rho, D}$. It follows that:

$$f^* \circ cw^{P, D} = cw^{f^* P, f^* D}.$$

The result then readily follows by Lemma 7.4. □

Proposition 7.8. *Let* $G \hookrightarrow Z \to Y$ *be an ordinary smooth principal* G*-bundle, and* ρ *as above. Let* Z^Δ *be the induced simplicial* G*-bundle over* $Y_\bullet$ *as in Example 5.6. Then:*

(1) *The form* ω^{ρ,D^Δ} *is the simplicial differential form induced by* $\omega^{\rho,D}$*, where induced is as in Example 4.4. In particular,*

$$cw^{Z^\Delta} \simeq \Theta \circ cw^Z,$$

where Θ *is as in* (4.3).

(2) *If* $c^\rho(Z) \in H^{2k}(Y,\mathbb{R})$ *is the Chern-Weil characteristic class as in* (7.5)*, then*

$$|c^\rho(Z^\Delta)|_{sm} = c^\rho(Z), \tag{7.6}$$

where $|c^\rho(Z^\Delta)|_{sm}$ *is as in part 2 of Notation 4.9.*

Proof. Fix a smooth G-connection D on Z. This induces a simplicial G-connection D^Δ on Z^Δ, as in Example 6.3. Let $\omega^{\rho,D}$ denote the smooth Chern-Weil differential $2k$-form on Y, as in (7.4).

Now,

$$\begin{aligned}\forall \Sigma : \omega^{\rho,D^\Delta}(\Sigma) &= \omega^{\rho,\widetilde{\Sigma}^* D} \text{ by definitions}\\ &= \Sigma^* \omega^{\rho,D} \text{ by standard naturality of Chern-Weil forms.}\end{aligned}$$

So we obtain the first part of the Proposition.

Let $\alpha^{\rho,D} = \int \omega^{\rho,D} \in C^{2k}(Y,\mathbb{R})$. It readily follows by Lemma 4.11 that:

$$|c^\rho(Z^\Delta)|_{sm} = (I^\vee)^{-1}([\alpha^{\rho,D^\Delta}]) = [\alpha^{\rho,D}] = c^\rho(Z),$$

where $I^\vee$ is as in (4.9). □

8. The universal simplicial G-bundle

Briefly, a Grothendieck universe is a set $\mathcal{U}$ forming a model for set theory. That is if we interpret all terms of set theory as elements of $\mathcal{U}$, then all the set theoretic constructions keep us within $\mathcal{U}$. We will assume Grothendieck's axiom of universes which says that for any (pure) set X there is a Grothendieck universe $\mathcal{U} \ni X$. Intuitively, such a universe $\mathcal{U}$ is formed by taking all possible set theoretic constructions starting with X. For example if $\mathcal{P}(X)$ denotes the power set of X, then $\mathcal{P}(X) \in \mathcal{U}$. Note that this axiom is beyond ZFC, and the resulting axiomatic system is sometimes denoted as $ZFCG$. This is now a common framework of modern set theory, especially in the context of category theory, c.f. [24]. In some contexts one works with universes implicitly. This is impossible here, as we need to establish certain universe independence.

Let G be a locally convex Lie group. Let $\mathcal{U}$ be a Grothendieck universe satisfying:

$$G \in \mathcal{U}, \quad \forall n \in \mathbb{N} : \Delta^n \in \mathcal{U},$$

where Δ^n are the usual topological n-simplices. Such a $\mathcal{U}$ will be called G-***admissible***.

We will construct smooth Kan complexes $BG^{\mathcal{U}}$ for each G-admissible $\mathcal{U}$. Moreover, we will construct a weak equivalence $|BG^{\mathcal{U}}| \to BG$ for each $\mathcal{U}$, where BG the standard Milnor classifying space.

If G has the homotopy type of a CW complex, then BG has the homotopy type of a CW complex. In particular, by Whitehead's theorem the homotopy type of $|BG^{\mathcal{U}}|$ is the independent of $\mathcal{U}$, and in fact $|BG^{\mathcal{U}}|$ is BG up to homotopy.

Definition 8.1. *A* $\mathcal{U}$**-small set** *is an element of* $\mathcal{U}$*. For* X *a smooth simplicial set, a smooth simplicial* G*-bundle* $P \to X$ *will be called* $\mathcal{U}$**-small** *if for each* n $X(n)$ *is* $\mathcal{U}$*-small, and for each simplex* Σ *of* X *the bundle* P_Σ *is* $\mathcal{U}$*-small.*

8.1. **The classifying spaces** $BG^{\mathcal{U}}$**.** Let $\mathcal{U}$ be G-admissible. We define a simplicial set $BG^{\mathcal{U}}$, whose set of k-simplices $BG^{\mathcal{U}}(k)$ is the set of $\mathcal{U}$-small smooth simplicial G-bundles over $\Delta^k_\bullet$. The simplicial maps are defined by pull-back so that given a map $i \in hom_\Delta(m,n)$ the map

$$BG^{\mathcal{U}}(i) : BG^{\mathcal{U}}(n) \to BG^{\mathcal{U}}(m)$$

is the natural pull-back:

$$BG^{\mathcal{U}}(i)(P) = i_\bullet^* P,$$

for $i_\bullet$, the induced map $i_\bullet : \Delta^m_\bullet \to \Delta^n_\bullet$, $P \in BG^{\mathcal{U}}(n)$ a simplicial G-bundle over $\Delta^n_\bullet$, and where the pull-back map $i^*_\bullet$ is as in Section 5.1. Then Lemma 5.13 ensures that $BG^{\mathcal{U}} : \Delta^{op} \to s-Set$ is a functor, so that we get a simplicial set $BG^{\mathcal{U}}$.

We define a smooth simplicial set structure g on $BG^{\mathcal{U}}$ as follows. Given a d-simplex $P \in BG^{\mathcal{U}}(d)$ the induced map

$$(g(P) = P_*) : \Delta^d_\bullet \to BG^{\mathcal{U}},$$

is defined naturally by

$$P_*(\sigma) := \sigma^*_\bullet P. \tag{8.1}$$

where P on the right is the initial simplicial G-bundle $P \to \Delta^d_\bullet$. More explicitly, $\sigma \in \Delta^d_\bullet(k)$ is a smooth map $\sigma : \Delta^k \to \Delta^d$, $\sigma_\bullet : \Delta^k_\bullet \to \Delta^d_\bullet$ denotes the induced map and the pull-back is as previously defined. We need to check the push-forward functoriality Axiom 2b.

Let $\sigma \in \Delta^d_\bullet(k)$, then for all $j \in \mathbb{N}, \rho \in \Delta^k_\bullet(j)$:

$$\begin{aligned}(P_*(\sigma))_*(\rho) &= (\sigma^*_\bullet P)_*(\rho) \\ &= \rho^*_\bullet(\sigma^*_\bullet P), \text{ by definition of } g.\end{aligned}$$

And

$$\begin{aligned}P_* \circ \sigma_\bullet(\rho) &= (\sigma_\bullet(\rho))^*_\bullet P \\ &= (\sigma_\bullet \circ \rho_\bullet)^* P, \text{ as } \sigma_\bullet \text{ is smooth} \\ &= \rho^*_\bullet(\sigma^*_\bullet P).\end{aligned}$$

And so

$$(P_*(\sigma))_* = P_* \circ \sigma_\bullet,$$

so that $BG^{\mathcal{U}}$ is indeed a smooth simplicial set.

8.2. **The universal smooth simplicial G-bundle $EG^{\mathcal{U}} \to BG^{\mathcal{U}}$.** To abbreviate already complex notation, in what follows V denotes $BG^{\mathcal{U}}$ for a general, G-admissible $\mathcal{U}$. There is a tautological functor

$$E : \Delta^{sm}(V) \to \mathcal{G} \tag{8.2}$$

that we now describe.

A smooth map $P : \Delta^d_\bullet \to V$, uniquely corresponds to a d-simplex P^b of V via Proposition 3.7, i.e. a simplicial G-bundle $P^b \to \Delta^d_\bullet$. In other words P^b is the bundle:

$$P^b = P(id^d), \tag{8.3}$$

for $id^d : \Delta^d \to \Delta^d$ the identity, and where the equality is an equality of simplicial G-bundles, in other words functors.

Notation 8.2. *Although we disambiguate in the discussion just below, later on we may conflate the notation P, P^b with just P.*

Recalling that P^b is a certain functor $\Delta^{sm}(\Delta^d_\bullet) \to \mathcal{G}$ we then set:

$$E(P) = P^b(id^d_\bullet).$$

We now define the action of E on morphisms. Suppose we have a morphism $m \in \Delta^{sm}(V)$:

$$\begin{array}{ccc} \Delta^k_\bullet & \xrightarrow{\widetilde{m}_\bullet} & \Delta^d_\bullet \\ & \searrow^{P_1} & \downarrow^{P_2} \\ & & V, \end{array}$$

then we have an equality:

$$\begin{aligned} P_1^b &= P_1(id^k) \text{ (8.3)} \\ &= (P_2 \circ \widetilde{m}_\bullet)(id^k) \\ &= P_2(\widetilde{m}) \\ &= (P_2^b)_*(\widetilde{m}), \text{ thinking of } P_2^b \text{ as a simplex of } V \\ &= P_2^b \circ \Delta^{sm}\widetilde{m}_\bullet, \text{ by (8.1)}. \end{aligned} \tag{8.4}$$

So that

$$P_1^b(id^k_\bullet) = P_2^b(\widetilde{m}_\bullet \circ id^k_\bullet) = P_2^b(\widetilde{m}_\bullet).$$

We have a tautological morphism $e_m \in \Delta^{sm}(\Delta^d_\bullet)$ corresponding to the diagram:

$$\begin{array}{ccc} \Delta^k_\bullet & \xrightarrow{\widetilde{m}_\bullet} & \Delta^d_\bullet \\ & \searrow^{\widetilde{m}_\bullet} & \downarrow^{id^d_\bullet} \\ & & \Delta^d_\bullet. \end{array}$$

So we get a smooth G-bundle map:

$$P_2^b(e_m) : (E(P_1) = P_2^b(\widetilde{m}_\bullet)) \to (E(P_2) = P_2^b(id^d_\bullet)),$$

which is over the smooth map $\widetilde{m} : \Delta^k \to \Delta^d$ induced by $\widetilde{m}_\bullet$. And we set $E(m) = P_2^b(e_m)$.

We need to check that with these assignments E is a functor. Suppose we have a diagram:

$$\begin{array}{ccccc} \Delta^l_\bullet & \xrightarrow{\widetilde{m}^0_\bullet} & \Delta^k_\bullet & \xrightarrow{\widetilde{m}^1_\bullet} & \Delta^d_\bullet \\ & & & \searrow^{P_1} & \downarrow P_2 \\ & & \xrightarrow{P_0} & & V. \end{array}$$

In other words, we have a diagram for the composition $m = m^1 \circ m^0$ in $\Delta^{sm}(V)$. Then $e_m = e_{m^1} \circ e'_{m^0}$ where e'_{m^0} is the diagram:

$$\begin{array}{ccc} \Delta^l_\bullet & \xrightarrow{\widetilde{m}^0_\bullet} & \Delta^k_\bullet \\ & \searrow^{\widetilde{m}_\bullet} & \downarrow \widetilde{m}^1_\bullet \\ & & \Delta^d_\bullet, \end{array}$$

and e_{m^1} is the diagram:

$$\begin{array}{ccc} \Delta^k_\bullet & \xrightarrow{\widetilde{m}^1_\bullet} & \Delta^d_\bullet \\ & \searrow^{\widetilde{m}^1_\bullet} & \downarrow id^d_\bullet \\ & & \Delta^d_\bullet. \end{array}$$

So

$$E(m) = P_2^b(e_m) = P_2^b(e_{m^1}) \circ P_2^b(e'_{m^0}) = E(m^1) \circ P_2^b(e'_{m^0}).$$

Now,

$$\begin{aligned} E(m_0) &= P_1^b(e_{m^0}) \\ &= (P_2^b \circ \Delta^{sm}\widetilde{m}^1_\bullet)(e_{m^0}), \text{ analogue of (8.4)} \\ &= P_2^b(e'_{m_0}). \end{aligned}$$

And so we get: $E(m) = E(m_1) \circ E(m_0)$. Thus, E is a functor.

By construction the functor E satisfies the compatibility condition, and hence determines a simplicial G-bundle.

Definition 8.3. *Given $G, \mathcal{U}$ as previously, the universal simplicial G-bundle $EG^{\mathcal{U}}$ is defined to be the functor E as constructed above.*

Proposition 8.4. *$BG^{\mathcal{U}}$ is a Kan complex.*

Proof. In what follows we again abbreviate $BG^{\mathcal{U}}$ as V. Recall that $\Lambda^n_k \subset \Delta^n_{simp}$, denotes the sub-simplicial set corresponding to the "boundary" of Δ^n with the k'th face removed, where by k'th face we mean the face opposite to the k'th vertex. Let $h : \Lambda^n_k \to V$, $0 \leq k \leq n$, be a simplicial map, this is also called a horn. We need to construct an extension of h to Δ^n_{simp}.

For simplicity we assume $n = 2$, and $k = 1$ as the general case is identical. There are three natural inclusions

$$i_j : \Delta^0_{simp} \to \Delta^2_{simp},$$

$j = 0, 1, 2$, with i_1 corresponding to the inclusion of the horn vertex. The corresponding 0-simplices will be denoted by $0, 1, 2$. Let

$$\sigma_{i,j} : \Delta^1_{simp} \to \Delta^2$$

be the edge between vertexes i, j, that is $\sigma_{i,j}(0) = i$, $\sigma_{i,j}(1) = j$.

Let us denote by L the smooth sub-simplicial set of $\Delta^2_\bullet$ corresponding to the simplices whose images lie in image $\sigma_{0,1}$ or image $\sigma_{1,2}$. There are then smooth maps $\sigma_{i,j} : \Delta^1_\bullet \to L$ extending $\sigma_{i,j}$ above.

The map h above, induces a smooth map $h_\bullet : L \to V$, and we denote $P := h^*_\bullet E$. So $P \to L$ is a simplicial G-bundle. The extension of h to Δ^2_{simp} will be accomplished once we extend the P over $\Delta^2_\bullet$.

Lemma 8.5. *The bundle $P \to L$ is trivializable.*

Proof. Set $P_{i,j} := \sigma^*_{i,j} P$, then by Theorem 5.10 $P_{i,j}$ is induced by a smooth G-bundle over Δ^1. The latter is trivializable, and hence $P_{i,j}$ is trivializable as a simplicial G-bundle. Denote by $\phi_{i,j} : \Delta^1_\bullet \times G \to P_{i,j}$ the corresponding trivialization over the $id : \Delta^1_\bullet \to \Delta^1_\bullet$.

Denoting $0, 1 \in \Delta^1_\bullet(0)$ the end-point vertices as previously, $\phi_{0,1}$ induces a G-equivariant smooth map:

$$G = (\Delta^1_\bullet \times G)(1) \to P_{0,1}(1),$$

and we denote this map by ϕ_1. Likewise, $\phi_{1,2}$ induces a smooth map:

$$G = (\Delta^1_\bullet \times G)(0) \to P_{1,2}(0),$$

and we denote this map by ϕ_2.

Now the map $\phi_1^{-1} \circ \phi_2$ may not be the $id : G \to G$, but clearly we may adjust $\phi_{1,2}$ so that it is. And so we may assume this holds.

Then $\phi_{0,1}$ and $\phi_{1,2}$ clearly induce a trivialization $tr : T \to P$, for $T \to L$ the trivial simplicial G-bundle.

□

We have the trivial extension of T to the trivial simplicial G-bundle over $\Delta^2_\bullet$. And so by the lemma above it should be clear that P likewise has an extension $\widetilde{P}$ over $\Delta^2_\bullet$, but we need this extension to be $\mathcal{U}$-small so that we must be explicit.

We proceed inductively. Let D^0 denote the full sub-category of $\Delta^{sm}(\Delta^2_\bullet)$ with the set objects $\operatorname{obj} L \cup \Delta^2_\bullet(0)$ (non-disjoint union).

We extend P to a functor $\widetilde{P}^0 : D^0 \to \mathcal{G}$. For $\sigma \in \Delta^2_\bullet(d)$, if σ has image in the horn $\Lambda^2_1 \subset \Delta^2$, then set $\widetilde{P}^0(\sigma) = P(\sigma)$. The extension of $\widetilde{P}^0$ to morphisms in D^0 is then taken to be the trivial extension.

Let $T^0 : D^0 \to \mathcal{G}$ be the trivial functor (as in the definition of a trivial bundle in Example 5.5). Then in addition, there is clearly a natural transformation $tr^0 : T^0 \to \widetilde{P}^0$ extending the natural transformation tr of the lemma above.

Let $S(n)$ be the statement:

(1) There is an extension $\widetilde{P}^n$ of P over the full-subcategory $D^n \subset \Delta^{sm}(\Delta^2_\bullet)$, defined analogously to D^0, with objects

$$\operatorname{obj} L \cup \bigcup_{0 \leq k \leq n} \Delta^2_\bullet(k).$$

(2) $\widetilde{P}^n$ satisfies compatibility.

(3) There is a natural transformation $tr^n : T^n \to \widetilde{P}^n$ extending the natural transformation tr of the lemma above. Where $T^n : D^n \to \mathcal{G}$ is the trivial functor defined analogously to T^0 above.

We prove

$$S(n) \implies S(n+1),$$

and that moreover the corresponding functor $\widetilde{P}^{n+1}$ can be chosen to extend $\widetilde{P}^n$. Then natural induction implies the existence of the needed extension $\widetilde{P}$ over $\Delta^2_\bullet$.

Let $\sigma \in \Delta^2_\bullet(n+1)$. By the hypothesis $S(n)$ the functor:

$$\sigma^*_\bullet \widetilde{P}^n : C_{n+1} \to \mathcal{G},$$

is defined, where C_{n+1} denotes the sub-category of $\Delta(\partial \Delta^{n+1}_{simp})$ with objects all non-degenerate objects and with morphisms injections. (The pull-back is as in Section 5.1).

We then have a topological bundle

$$p' : N'_\sigma \to \partial \Delta^{n+1}$$

defined as the colimit of $\sigma^*_\bullet \widetilde{P}^n$ over C_{n+1}.

Let

$$inc : \partial \Delta^{n+1} \to \Delta^{n+1}$$

denote the inclusion. We first construct a principal G-bundle with discrete topology

$$N_\sigma \xrightarrow{p} \Delta^{n+1},$$

by the following conditions:

$$N_\sigma|_{\partial \Delta^{n+1}} := p^{-1}(\partial \Delta^{n+1}) = N'_\sigma, \tag{8.5}$$

$$N_\sigma|_{(\Delta^{n+1})^\circ} = (\Delta^{n+1})^\circ \times G, \tag{8.6}$$

where the projection map p is determined by the maps $p' : N'_\sigma \to \partial \Delta^{n+1}$, and the projection map $(\Delta^{n+1})^\circ \times G \to (\Delta^{n+1})^\circ$.

By the inductive hypothesis $S(n)$ part 3, there is a distinguished trivialization $h_n : \partial \Delta^{n+1} \times G \to N'_\sigma$, corresponding to tr^n. The map h_n and the identity map $(\Delta^{n+1})^\circ \times G \to (\Delta^{n+1})^\circ$ induce a discrete G-bundle isomorphism $\Delta^{n+1} \times G \to N_\sigma$. Then push-forward the smooth G-bundle structure and the topology along this map. The resulting smooth G-bundle is then set to be $\widetilde{P}^{n+1}(\sigma)$.

We have thus defined the extension $\widetilde{P}^{n+1}$ on objects. We now need to treat morphisms in $D^{n+1} \subset \Delta(\Delta^2_\bullet)$. For any d-simplex ρ of Δ^2 let ρ_i denote the i'th face of ρ,

$0 \leq i \leq d$. For clarity, this means that we have an inclusion morphism $inc_i : \rho_i \to \rho$ corresponding to the diagram:

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\widetilde{inc_i}} & \Delta^d \\ & \searrow^{\rho_i} & \downarrow \rho \\ & & \Delta^2, \end{array}$$

where $\widetilde{inc_i}$ is the topological face inclusion map corresponding to the face opposite the vertex i, also called the i-face of Δ^d.

By construction we have natural maps:

$$\widetilde{P}^{n+1}(inc_i) : \widetilde{P}^n(\sigma_i) = \widetilde{P}^{n+1}(\sigma_i) \to \widetilde{P}^{n+1}(\sigma),$$

for each $i \in \{0, \ldots, n+1\}$.

If $m : \sigma^{n+1} \to \rho^{n+1}$ is an identity morphism, then set $\widetilde{P}^{n+1}(m)$ to be the id.

Suppose we given a morphism $m : \sigma \to \rho$, $\sigma \in \Delta^2_\bullet(n+1)$, $\rho \in \Delta^2_\bullet(n)$. We then define $\widetilde{P}^{n+1}(m)$ as follows. First we define a map:

$$\partial : inc^* \widetilde{P}^{n+1}(\sigma) \to \widetilde{P}^{n+1}(\rho),$$

for $inc : \partial\Delta^{n+1} \to \Delta^{n+1}$ the inclusion.

Over a j-face of Δ^{n+1}, $\widetilde{P}^{n+1}(\sigma)$ is naturally identified with $\widetilde{P}^n(\sigma_j)$. The composition $m_j = m \circ inc_j$,

$$\sigma_j \xrightarrow{inc_j} \sigma \xrightarrow{m} \rho,$$

is a morphism in D^n. So we have the map

$$\widetilde{P}^n(m_j) : \widetilde{P}^n(\sigma_j) \to \widetilde{P}^{n+1}(\rho).$$

The collection of these maps for $0 \leq j \leq n+1$ then naturally induces the map ∂.

We then define $\widetilde{P}^{n+1}(m)$ using the identifications (8.5), (8.6) as follows. Set $\widetilde{P}^{n+1}(m)$ to be the map ∂ on

$$\widetilde{P}^{n+1}(\sigma)|_{\partial\Delta^{n+1}} = p^{-1}(\partial\Delta^{n+1}).$$

The hypothesis $S(n)$ part 3 ensures that $\widetilde{P}^{n+1}(m)$ has an extension to a map $\widetilde{P}^{n+1}(\sigma) \to \widetilde{P}^{n+1}(\rho)$.

As $\widetilde{P}^{n+1}$ must extend $\widetilde{P}^n$, combined with the construction above, we have thus specified the functor $\widetilde{P}^{n+1}$ on a generating set of morphisms in D^{n+1}, which defines $\widetilde{P}^{n+1}$ completely. For example, given a degeneracy $m : \sigma^n \to \rho^{n-2}$, we may factorize it as $\sigma^n \xrightarrow{m'} \rho^{n-1} \xrightarrow{pr} \rho^{n-2}$, and then set:

$$\widetilde{P}^{n+1}(m) = (\widetilde{P}^{n+1}(pr) = \widetilde{P}^n(pr)) \circ \widetilde{P}^{n+1}(m').$$

Functoriality of $\widetilde{P}^n$, implies that this is well defined. And by construction $\widetilde{P}^{n+1}$ will be a functor satisfying compatibility. So we are done.

$\square$

Theorem 8.6. *Let X be a smooth simplicial set. $\mathcal{U}$-small simplicial G-bundles $P \to X$ are "classified by" smooth maps*

$$f_P : X \to BG^{\mathcal{U}}.$$

Specifically:

(1) *For every $\mathcal{U}$-small P there is a natural smooth map $f_P : X \to BG^{\mathcal{U}}$ so that*
$$f_P^* EG^{\mathcal{U}} = P$$
as simplicial G-bundles. We say in this case that f_P **classifies** *P.*

(2) *If P_1, P_2 are isomorphic $\mathcal{U}$-small smooth simplicial G-bundles over X then the classifying maps f_{P_1}, f_{P_2} are smoothly homotopic, as in Definition 3.14.*

(3) *If $X = Y_\bullet$ for Y a smooth manifold and $f, g : X \to BG^{\mathcal{U}}$ are smoothly homotopic then $P_f = f^* EG^{\mathcal{U}}, P_g = g^* EG^{\mathcal{U}}$ are isomorphic simplicial G-bundles.*

Note that the above is a partly stronger (because of equality in Part 1) and partly weaker than just saying that isomorphism classes of $\mathcal{U}$-small bundles over X are in correspondence with smooth homotopy classes of maps $X \to BG^{\mathcal{U}}$. It is strictly stronger when $X = Y_\bullet$ for Y a smooth manifold.

Proof. Set $V = BG^{\mathcal{U}}$, $E = EG^{\mathcal{U}}$. Let $P \to X$ be a $\mathcal{U}$-small simplicial G-bundle. Define $f_P : X \to V$ by:

$$f_P(\Sigma) = \Sigma_*^* P, \tag{8.7}$$

where $\Sigma \in \Delta^d(X)$, $\Sigma_* : \Delta^d_\bullet \to X$, the induced map, and the pull-back $\Sigma_*^* P$ our usual simplicial G-bundle pull-back. We check that the map f_P is simplicial.

Let $m : k \to d$ be a morphism in Δ. We need to check that the following diagram commutes:

$$\begin{array}{ccc} X(d) & \xrightarrow{X(m)} & X(k) \\ \downarrow{\scriptstyle f_P} & & \downarrow{\scriptstyle f_P} \\ V(d) & \xrightarrow{V(m)} & V(k). \end{array}$$

Let $\Sigma \in X(d)$, then by push-forward functoriality Axiom 2b $(X(m)(\Sigma))_* = \Sigma_* \circ m_\bullet$ where $m_\bullet : \Delta^k_\bullet \to \Delta^d_\bullet$ is the simplicial map induced by $m : \Delta^k \to \Delta^m$. And so

$$f_P(X(m)(\Sigma)) = (\Sigma_* \circ m_\bullet)^* P = m_\bullet^*(\Sigma_*^* P) = V(m)(f_P(\Sigma)),$$

where the second equality uses Lemma 5.13. Hence the diagram commutes.

We now check that f_P is smooth. Let $\Sigma \in X(d)$, then we have:

$$\begin{aligned} (f_P(\Sigma))_*(\sigma) &= \sigma_\bullet^*(\Sigma_*^* P) \\ &= (\Sigma_* \circ \sigma_\bullet)^* P, \quad \text{Lemma 5.13} \\ &= (\Sigma_*(\sigma))_*^* P, \quad \text{as } \Sigma_* \text{ is smooth, Lemma 3.6} \\ &= f_P(\Sigma_*(\sigma)), \quad \text{by (8.7)} \\ &= (f_P \circ \Sigma_*)(\sigma), \end{aligned}$$

and so f_P is smooth.

We check that $f_P^* E = P$. Let $\Sigma : \Delta^d_\bullet \to X$ be smooth, and $\sigma \in \Delta^d_\bullet$. First, we need the identity:

$$\begin{aligned}\Delta^{sm} f_P(\Sigma)(\sigma) = (f_P \circ \Sigma)(\sigma) = f_P(\Sigma(\sigma)) &= (\Sigma(\sigma))^*_* P \text{ by definition of } f_P \\ &= (\Sigma^* \circ \sigma_\bullet)^* P \text{ as } \Sigma \text{ is smooth} \\ &= \sigma^*_\bullet(\Sigma^* P) \text{ Lemma 5.13} \\ &= g(\Sigma^* P)(\sigma).\end{aligned} \tag{8.8}$$

So

$$\Delta^{sm} f_P(\Sigma) = g(\Sigma^* P). \tag{8.9}$$

Then

$$\begin{aligned}f_P^* E(\Sigma) = (E \circ \Delta^{sm} f_P)(\Sigma) &= E(g(\Sigma^* P)), \text{ by (8.8)} \\ &= (\Sigma^* P)(id^d_\bullet), \text{ definition of } E \\ &= P(\Sigma).\end{aligned}$$

So $f_P^* E = P$ on objects.

Now let m be a morphism:

$$\begin{array}{ccc} \Delta^k_\bullet & \xrightarrow{\widetilde{m}_\bullet} & \Delta^d_\bullet \\ & \searrow^{\Sigma_1} & \downarrow{\scriptstyle \Sigma_2} \\ & & X, \end{array}$$

in $\Delta^{sm}(X)$. We then have, for e_m is as in the definition of E:

$$\begin{aligned}f_P^* E(m) &= E(\Delta^{sm} f_P(m)) \\ &= (\Delta^{sm} f_P(\Sigma_2))^b(e_m) \text{ by definition of } E \\ &= \Sigma_2^* P(e_m) \text{ by (8.9)} \\ &= (P \circ \Delta^{sm} \Sigma_2)(e_m).\end{aligned}$$

But $\Delta^{sm} \Sigma_2(e_m)$ is the diagram:

$$\begin{array}{ccc} \Delta^k_\bullet & \xrightarrow{\widetilde{m}_\bullet} & \Delta^d_\bullet \\ & \searrow^{\Sigma_2 \circ \widetilde{m}_\bullet} & \downarrow{\scriptstyle \Sigma_2 \circ id^d_\bullet} \\ & & X, \end{array}$$

i.e. it is the diagram m. So $(P \circ \Delta^{sm} \Sigma_2)(e_m) = P(m)$. Thus, $f_P^* E = P$ on morphisms.

So we have proved the first part. We now prove the second part. Suppose that $\phi : P_1 \to P_2$ is an isomorphism of $\mathcal{U}$-small simplicial G-bundles over X. We construct a $\mathcal{U}$-small simplicial G-bundle $\widetilde{P}$ over $X \times I$ as follows, where $I = \Delta^1_\bullet$ as before.

Let σ be a k-simplex of X. Then ϕ specifies a G-bundle diffeomorphism $\phi_\sigma : P_1(\sigma) \to P_2(\sigma)$ over the identity map $\Delta^k \to \Delta^k$. Let M_σ be the mapping cylinder of ϕ_σ. So that

$$M_\sigma = (P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma))/\sim, \tag{8.10}$$

for $\sim$ the equivalence relation generated by the condition

$$(x,1) \in P_1(\sigma) \times \Delta^1 \sim \phi(x) \in P_2(\sigma).$$

Then M_σ is a smooth G-bundle over $\Delta^k \times \Delta^1$.

Let pr_X, pr_I be the natural projections of $X \times I$, to X respectively I. Let Σ be a d-simplex of $X \times I$, for any d. Set $\sigma_1 = pr_X \Sigma$, and $\sigma_2 = pr_I(\Sigma)$. Let $id^d : \Delta^d \to \Delta^d$ be the identity, so

$$(id^d, \sigma_2) : \Delta^d \to \Delta^d \times \Delta^1,$$

is a smooth map, where σ_2 is the corresponding smooth map $\sigma_2 : \Delta^d \to \Delta^1 = [0,1]$. We then define

$$\widetilde{P}_\Sigma := (id^d, \sigma_2)^* M_{\sigma_1},$$

which is a smooth G-bundle over Δ^d.

Notice that if Σ is in $X \times 0_\bullet \subset X \times I$, then we do *not* have $\widetilde{P}(\Sigma) = P_1(\Sigma)$, instead there is a natural isomorphism. This is for the same reason that fixing the standard construction of the set theoretic pull-back, a bundle $P \to B$ is not set theoretically equal to the bundle $id^* P \to B$, for $id : B \to B$ the identity, (but they are of course naturally isomorphic.) However, we can adjust the construction of $\widetilde{P}_\Sigma$ so that $\widetilde{P}(\Sigma) = P_1(\Sigma)$ does hold, similarly to the inductive procedure in the proof of Proposition 8.4. In what follows, we ignore this minor ambiguity.

Suppose that $\rho : \sigma \to \sigma'$ is a morphism in $\Delta^{sm}(X)$, for σ a k-simplex and σ' a d-simplex. As ϕ is a simplicial G-bundle map, we have a commutative diagram:

$$\begin{array}{ccc} P_1(\sigma) & \xrightarrow{P_1(\rho)} & P_1(\sigma') \\ \downarrow{\phi_\sigma} & & \downarrow{\phi_{\sigma'}} \\ P_2(\sigma) & \xrightarrow{P_2(\rho)} & P_2(\sigma'). \end{array} \tag{8.11}$$

And so we get a naturally induced (by the pair of maps $P_1(\rho), P_2(\rho)$) bundle map:

$$\begin{array}{ccc} M_\sigma & \xrightarrow{g_\rho} & M_{\sigma'} \\ \downarrow & & \downarrow \\ \Delta^k \times \Delta^1 & \xrightarrow{\widetilde{\rho} \times id} & \Delta^d \times \Delta^1 . \end{array} \tag{8.12}$$

More explicitly, let $q_\sigma : P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma) \to M_\sigma$ denote the quotient map. Define

$$\widetilde{g}_\rho : P_1(\sigma) \times \Delta^1 \sqcup P_2(\sigma) \to M_{\sigma'}$$

by:

$$\widetilde{g}(x,t) = q_{\sigma'}((P_1(\rho)(x), t)) \in M_{\sigma'},$$

for

$$(x,t) \in P_1(\sigma) \times \Delta^1,$$

while $\widetilde{g}_\rho(y) = q_{\sigma'}(P_2(\rho)(y))$ for $y \in P_2(\sigma)$. By commutativity of (8.11) $\widetilde{g}_\rho$ induces the map $g_\rho : M_\sigma \to M_{\sigma'}$, appearing in (8.12).

Now suppose we have a morphism $m : \Sigma \to \Sigma'$ in $\Delta^{sm}(X \times I)$, where Σ is a k-simplex and Σ' is a d-simplex. Then we have a commutative diagram:

$$
\begin{array}{ccc}
M_\sigma & \xrightarrow{g_{pr_X}(m)} & M_{\sigma'} \\
\downarrow & & \downarrow \\
\Delta^k \times \Delta^1 & \xrightarrow{\widetilde{m} \times id} & \Delta^d \times \Delta^1 \\
{\scriptstyle h_1}\uparrow & & {\scriptstyle h_2}\uparrow \\
\Delta^k & \xrightarrow{\widetilde{m}} & \Delta^d \\
\uparrow & & \uparrow \\
\widetilde{P}_\Sigma & & \widetilde{P}_{\Sigma'}
\end{array}
\tag{8.13}
$$

where $h_1 = (id^k, pr_I(\Sigma))$ and $h_2 = (id^d, pr_I(\Sigma'))$. We then readily get an induced natural bundle map:

$$\widetilde{P}(m) : \widetilde{P}_\Sigma \to \widetilde{P}_{\Sigma'},$$

as left most and right most arrows in the above commutative diagram are the natural maps in pull-back squares, and so by universality of the pull-back such a map exists and is uniquely determined. Of course $\widetilde{P}(m)$ is the unique map making the whole diagram (8.13) commute.

With the above assignments, it is immediate that $\widetilde{P}$ is indeed a functor, by the uniqueness of the assignment $\widetilde{P}(m)$. And this determines our $\mathcal{U}$-small smooth simplicial G-bundle $\widetilde{P} \to X \times I$. By the first part of the theorem, we have an induced smooth classifying map $f_{\widetilde{P}} : X \times I \to V$. By construction, it is a homotopy between f_{P_1}, f_{P_2}. So we have verified the second part of the theorem.

We now prove the third part of the theorem. Let $X = Y_\bullet$. Suppose that $f, g : X \to V$ are smoothly homotopic, and let $H : X \times I \to V$ be the corresponding smooth homotopy. Now $P_H = H^*E$ is a simplicial G-bundle over $X \times I = (Y \times [0,1])_\bullet$ and hence by Theorem 5.10 P_H is induced by a smooth G-bundle P'_H over $Y \times [0,1]$.

Now by construction

$$P_f \simeq (P'_H|_{Y \times \{0\}})^\Delta$$

and

$$P_g \simeq (P'_H|_{Y \times \{1\}})^\Delta.$$

And

$$P'_H|_{Y \times \{0\}} \simeq P'_H|_{Y \times \{1\}}$$

by standard smooth bundle theory and hence

$$(P'_H|_{Y \times \{0\}})^\Delta \simeq (P'_H|_{Y \times \{1\}})^\Delta.$$

And so $P_f \simeq P_g$.

□

Since Theorem 5.10 works for manifolds with corners, and since $\Delta^k_\bullet \times I \simeq (\Delta^k \times \Delta^1)_\bullet$ the proof of the theorem above readily extends to give the following theorem. We say that a smooth G-bundle P over Δ^k is trivial over $\partial \Delta^k$, if there is a distinguished trivialization of P over $\partial \Delta^k$. A ***relative isomorphism*** of P_0, P_1 as above, is an isomorphism that is trivial trivial over $\partial \Delta^k$, in the respective distinguished trivializations.

Let $v_0 \in BG^{\mathcal{U}}(0)$ correspond to the trivial simplicial G-bundle $G \times \Delta^0_\bullet \to \Delta^0_\bullet$.

Theorem 8.7. *The set $\pi^{sm}_k(BG^{\mathcal{U}}, v_0)$ (Definition 3.15) is naturally isomorphic to the set $\mathcal{P}^{\mathcal{U}}_k$ of equivalence classes of smooth, $\mathcal{U}$-small G-bundles P over Δ^k trivial over $\partial \Delta^k$, where $P_0 \sim P_1$ if there is a relative bundle isomorphism from P_0 to P_1. The map*

$$cl_k : \pi^{sm}_k(BG^{\mathcal{U}}, v_0) \to \mathcal{P}^{\mathcal{U}}_k \tag{8.14}$$

is given by $[f] \mapsto [P_f]$, where $P_f = f^* EG^{\mathcal{U}}(id^k)$, $id^k : \Delta^k \to \Delta^k$ the identity.

We now study the dependence on a Grothendieck universe $\mathcal{U}$.

Theorem 8.8. *Let G be a locally convex Lie group. Let $\mathcal{U}$ be a G-admissible universe, let $|BG^{\mathcal{U}}|$ denote the geometric realization of $BG^{\mathcal{U}}$ and let BG^{top} denote the classifying space of G as defined by the Milnor construction [32]. Then there is a weak homotopy equivalence*

$$e^{\mathcal{U}} : |BG^{\mathcal{U}}| \to BG^{top},$$

which is natural in the sense that if $\mathcal{U} \in \mathcal{U}'$ then

$$[e^{\mathcal{U}'} \circ |i^{\mathcal{U},\mathcal{U}'}|] = [e^{\mathcal{U}}], \tag{8.15}$$

where $|i^{\mathcal{U},\mathcal{U}'}| : |BG^{\mathcal{U}}| \to |BG^{\mathcal{U}'}|$ is the map of geometric realizations, induced by the natural inclusion $i^{\mathcal{U},\mathcal{U}'} : BG^{\mathcal{U}} \to BG^{\mathcal{U}'}$ and where $[\cdot]$ denotes the homotopy class. In particular, if G has the homotopy type of a CW complex, then for all $\mathcal{U}$ $BG^{\mathcal{U}}$ has the homotopy type of BG^{top}.

Proof. To cut down on notation set $V := BG^{\mathcal{U}}$, and $E := EG^{\mathcal{U}}$, $v_0 \in V$ will be as above, and we will not disambiguate by decoration with $\mathcal{U}$.

Lemma 8.9. *Let $P \to X$ be a simplicial G-bundle. Set $|P| = \operatorname{colim}_{\Delta(X)} P$, where the colimit is understood to be in the category of topological G-bundles, and recalling that P is a functor $\Delta^{sm}(X) \to \mathcal{G}$, and so restricts to a functor $\Delta(X) \to \mathcal{G}$. Then the natural map $|P| \to |X|$ is a topological principal G-bundle. We call this the* **geometric realization of** *P.*

Proof. Set $P' = \mathcal{F}_1 \circ P$, where $\mathcal{F}_1$ is as in Definition 5.3. So there is a natural transformation $N : P \to P'$ of Top valued functors. By naturality of the colimit there is an induced map:

$$\operatorname{colim}_{\Delta(X)} P \to \operatorname{colim}_{\Delta(X)} P',$$

i.e. a continuous map $|P| \to |X|$.

We check that the map $|p|$ is a locally trivial fibration. By a topological d-simplex $\Sigma : \Delta^d \to |X|$ we shall mean the natural map (in the colimit diagram) $\Delta^d \to |X|$

corresponding to a non-degenerate d-simplex of X. If $v \in |X|$ is a vertex, i.e. the image of a topological 0-simplex $\Delta^0 \to |X|$, we construct a contractible open neighborhood $U_v \ni v$ as follows.

Let S_d be the set of the topological d-simplices with images containing v. For $\Sigma \in S_d$, let $\widetilde{\Sigma} = \operatorname{image} \Sigma - \operatorname{image} \Sigma_v$, where $\Sigma_v : \Delta^{d-1} \to |X|$ is the face of Σ not containing v. Then define:

$$U_v = \bigcup_d \bigcup_{\Sigma \in S_d} \widetilde{\Sigma}.$$

Over each $\widetilde{\Sigma}$ the bundle $|p|$ is obviously a trivializeable topological G-bundle. Moreover, any trivialization over the boundary of $\widetilde{\Sigma}$ (the collection of the remaining faces of Σ) may be extended to a trivialization over $\widetilde{\Sigma}$. We may then proceed inductively.

Set

$$U^k = \bigcup_{d \leq k} \bigcup_{\Sigma \in S_d} \widetilde{\Sigma},$$

by convention set $U^0 = \{v\}$. We construct G-bundle trivialization maps $\forall k \in \mathbb{N}$:

$$f^k : |p|^{-1}(U^k) \to U^k \times G,$$

with the property that each f^{k+1} extends f^k. Clearly, f^0 exists. Suppose we have constructed $f^0, \ldots, f^k$ for some $k \geq 0$ with the property above. We then construct f^{k+1}. Let $\Sigma \in S_{k+1}$, then over the boundary of $\widetilde{\Sigma}$ we already have a trivialization determined by f^k. Clearly the set J_Σ of extensions of this trivialization over $\widetilde{\Sigma}$ is nonempty.

Using axiom of choice fix an element of J_Σ for each $\Sigma \in S_{k+1}$, and this determines our extension f^{k+1}. [9] □

By the lemma above we have a topological G-bundle

$$|E| \to |V|.$$

Then

$$|E| \simeq e^* EG^{top}, \tag{8.16}$$

where

- EG^{top} is the universal G-bundle over BG^{top}.
- $$e = e^{\mathcal{U}} : |V| \to BG^{top}$$
  is uniquely determined up to homotopy.
- $\simeq$ here and the rest of this argument will mean G-bundle isomorphism or simplicial G-bundle isomorphism, depending on context.

[9] Strictly speaking, to formalize the existence of the infinite sequence $\{f^k\}$ (not just any fragment) requires a separate invocation of the axiom of choice, or more specifically the so called axiom of dependent choice, but this is standard.

We say that e ***classifies*** $|E| \to |V|$. We will show that e induces an isomorphism of all homotopy groups.

We first prove an auxiliary lemma. Let $\mathcal{U}'$ be a universe enlargement of $\mathcal{U}$, that is $\mathcal{U}'$ is a universe with $\mathcal{U} \in \mathcal{U}'$. There is a natural inclusion map

$$i = i^{\mathcal{U},\mathcal{U}'} : BG^{\mathcal{U}} \to BG^{\mathcal{U}'},$$

and

$$i^* EG^{\mathcal{U}'} = E\,^{10}.$$

Lemma 8.10. *Let G be a locally convex Lie group, then*

$$i_* : \pi_k^{sm}(BG^{\mathcal{U}}, v_0) \to \pi_k^{sm}(BG^{\mathcal{U}'}, v_0)$$

is a set isomorphism for all $k \in \mathbb{N}$.

Proof. We show that i_* is injective. Let's abbreviate $V = BG^{\mathcal{U}}$, $V' = BG^{\mathcal{U}'}$, $E = EG^{\mathcal{U}}$, $E' = EG^{\mathcal{U}'}$. Let $f, g : \Delta^k_\bullet \to V$ be a pair of smooth maps relative to v_0. Let P_f, P_g denote the smooth bundles over Δ^k as given by the correspondence of Theorem 8.7. Set $f' = i \circ f$, $g' = i \circ g$ and suppose that f', g' are smoothly homotopic. Then by Theorem 8.7, $P_{f'}, P_{g'}$ are relatively isomorphic and so $P_{f'}$ and $P_{g'}$ are relatively isomorphic. (Since we may identify $P_{f'}, P_{g'}$ with P_f, P_g.)

We now show surjectivity of i_*. Let $f : \Delta^k_\bullet \to V'$ be a smooth relative to v_0 map. Let $P_f \to \Delta^k$ be the corresponding smooth bundle trivial over $\partial\Delta^k$, via Theorem 8.7.

It is elementary that any such smooth bundle is isomorphic to a bundle obtained by the clutching construction corresponding to some smooth relative to e map $\phi : \Delta^{k-1} \to G$ (meaning as before the boundary is taken to e). Specifically, P' is isomorphic as a smooth G-bundle to a bundle of the form:

$$C_\phi = \Delta^k_- \times G \sqcup \Delta^k_+ \times G / \sim,$$

where:

(1) Δ^k_+, Δ^k_- are connected and closed subsets of Δ^k diffeomorphic to Δ^k, covering Δ^k, whose intersection is the image of a smooth embedding $i : \Delta^{k-1} \to \Delta^k$ mapping boundary to boundary, and mapping the interior to the interior of Δ^k.

(2) $\sim$ is the equivalence relation generated by the relation: for $(i(x), g) \in \Delta^k_- \times G$,
$$(i(x), g) \sim (i(x), \phi(x) \cdot g) \in D^k_+ \times G.$$

This gluing construction can be carried out in $\mathcal{L}$, for any G-admissible Grothendieck universe $\mathcal{L}$. In particular, C_ϕ is $\mathcal{U}$-small and C_ϕ and P_f are relatively isomorphic $\mathcal{U}'$-small smooth G-bundles.

If we denote by f_ϕ a representative for $cl_k^{-1}([C_\phi])$, then by Theorem 8.7 $f_\phi : \Delta^k_\bullet \to V'$ is smoothly relatively homotopic to f. But C_ϕ is $\mathcal{U}$-small, and hence $[C_\phi] \in \mathcal{P}^{\mathcal{U}}_k$ is $cl_k([f'])$ for a smooth based map $f' : \Delta^k_\bullet \to V$. It is immediate that $[i \circ f'] = [f_\phi]$, since $i^* E' = E$. And so $i_*([f']) = [f]$. □

[10] This is indeed an equality, not just a natural isomorphism.

Corollary 8.11. *Let G be a locally convex Lie group, and let $\mathcal{U}, \mathcal{U}'$ be as in the previous lemma. Then the natural map*

$$j : \mathcal{P}_k^{\mathcal{U}} \to \mathcal{P}_k^{\mathcal{U}'},$$

is a set bijection.

We now prove that $e_* : \pi_k(|V|, v_0) \to \pi_k(BG^{top}, x_0)$ is an isomorphism, where x_0 is any fixed point s.t. $EG^{\mathcal{U}}$ is given a fixed trivialization over x_0.

Let $f : \Delta^k \to BG^{top}$ be a continuous based at x_0 map. By Müller-Wockel [33, Theorem II.11], the bundle $P_f := f^* EG^{top}$ is topologically relatively isomorphic to a smooth G-bundle $P' \to \Delta^k$ trivial over the boundary. By the axiom of universes P' is $\mathcal{U}_0$-small for some G-admissible $\mathcal{U}_0 \ni \mathcal{U}$.

By Theorem 8.7 $[P'] = j(cl_k([g]))$ for some smooth relative to v_0

$$g : \Delta^k_\bullet \to V.$$

Hence P_f is relatively isomorphic as a topological G-bundle to $|g|^*|E|$, where

$$|g| : \Delta^k \simeq |\Delta^k_{simp}| \to |V|$$

is the natural map induced by g, and $\Delta^k \simeq |\Delta^k_{simp}|$ is the natural topological homeomorphism. And so P_f is relatively isomorphic to $|g|^* e^* EG^{top}$, and so as relative classes $[f] = e_*[g]$, so that we are done with surjectivity.

We now prove injectivity. Let $f_0, f_1 : \Delta^k \to |V|$ be continuous and relative to v_0. We may represent the relative classes $[f_i]$ by $|g_i|$, where

$$|g_i| : |\Delta^k_{simp}| \to |V|$$

are induced by some relative to v_0 maps $g_i : \Delta^k_{simp} \to V$. This is by generalities of homotopy groups of Kan complexes, see for instance [Chapter 1][16].

Let $P_i \to \Delta^k$ be the smooth trivial over the boundary G-bundles corresponding to g_i, meaning: take the simplices corresponding to g_i, these represent simplicial G-bundles over $\Delta^k_\bullet$ by construction of V, then evaluate on $id^k : \Delta^k \to \Delta^k$.

Now suppose that $[e \circ f_0] = [e \circ f_1]$. Then P_i are relatively isomorphic as topological G-bundles, and so by [33, Theorem II.12] P_i are smoothly relatively isomorphic G-bundles over Δ^k. And so by Theorem 8.7 g_i are smoothly homotopic. Consequently, $|g_i|$ are homotopic and so $[f_0] = [f_1]$.

It follows that:

$$\forall k \in \mathbb{N} : e_* : \pi_k(|V|, v_0) \to \pi_k(BG^{top}, x_0) \tag{8.17}$$

is an isomorphism.

Finally, we show naturality. Let

$$|i^{\mathcal{U},\mathcal{U}'}| : |V| \to |V'|$$

denote the map induced by the inclusion $i^{\mathcal{U},\mathcal{U}'}$. Since $E = (i^{\mathcal{U},\mathcal{U}'})^* E'$, we have that

$$|E| \simeq |i^{\mathcal{U},\mathcal{U}'}|^* |E'|$$

and so

$$|E| \simeq |i^{\mathcal{U},\mathcal{U}'}|^* \circ (e^{\mathcal{U}'})^* EG^{top},$$

by (8.16), from which the conclusion immediately follows. And we are done with the proof of the theorem. □

9. The universal Chern-Weil homomorphism

In this section G is a generalized Lie group and $\mathfrak{g}$ its Lie algebra. Pick any simplicial G-connection D on $EG^{\mathcal{U}} \to BG^{\mathcal{U}}$. By construction of Section 7 we obtain a dg homomorphism:

$$cw^D := cw^{EG^{\mathcal{U}},D} : \mathcal{I}(G) \to \Omega^\bullet(BG^{\mathcal{U}}, \mathbb{R}). \tag{9.1}$$

Let cw represent $[cw^D]$ as in Notation 7.6. This satisfies the following property:

Proposition 9.1. *Let $\mathcal{U}$ be a G-admissible Grothendieck universe. Let $P \to X$ be a $\mathcal{U}$-small simplicial G-bundle and let*

$$cw^P : \mathcal{I}(G) \to \Omega^\bullet(X, \mathbb{R}),$$

be as in Notation 7.6. Then

$$f_P^* \circ cw \simeq cw^P,$$

where $f_P : X \to BG^{\mathcal{U}}$ is the classifying map.

Proof. This follows immediately from Lemma 7.7. □

Let $e^{\mathcal{U}}$ be as in Theorem 8.8, then this is a weak equivalence. And so induces an isomorphism

$$e^{\mathcal{U}}_* : H^\bullet(|BG^{\mathcal{U}}|, \mathbb{R}) \to H^\bullet(BG^{top}, \mathbb{R}),$$

Hatcher [13, Proposition 4.21].

Set

$$c^{\rho,\mathcal{U}} := c^\rho(EG^{\mathcal{U}}) = [\int (cw(\rho))] \in H^{2k}(BG^{\mathcal{U}}, \mathbb{R}).$$

Then we define the cohomology class

$$c^\rho := e^{\mathcal{U}}_*(|c^{\rho,\mathcal{U}}|) \in H^{2k}(BG^{top}, \mathbb{R}),$$

where the G-admissible universe $\mathcal{U}$ is chosen arbitrarily and where

$$|c^{\rho,\mathcal{U}}| \in H^{2k}(|BG^{\mathcal{U}}|, \mathbb{R})$$

is as in Notation 4.9.

Lemma 9.2. *The cohomology class c^ρ is well-defined.*

Proof. Given another choice of a G-admissible universe $\mathcal{U}'$, let $\mathcal{U}'' \supset \{\mathcal{U}, \mathcal{U}'\}$ be a common universe enlargement. By Lemma 7.7 and Lemma 4.10

$$|i^{\mathcal{U},\mathcal{U}''}|^*(|c^{\rho,\mathcal{U}''}|) = |c^{\rho,\mathcal{U}}|.$$

Now $|i^{\mathcal{U},\mathcal{U}''}|$ is a weak equivalence by Theorem 8.8. Let

$$|i^{\mathcal{U},\mathcal{U}''}|^* : H^\bullet(|BG^{\mathcal{U}'}|, \mathbb{R}) \to H^\bullet(|BG^{\mathcal{U}}|, \mathbb{R})$$

be the corresponding algebra isomorphism, and let $|i^{\mathcal{U},\mathcal{U}''}|_*$ denote its inverse.

Then we have:

$$|i^{\mathcal{U},\mathcal{U}''}|_*(|c^{\rho,\mathcal{U}}|) = |c^{\rho,\mathcal{U}''}|. \tag{9.2}$$

Consequently,

$$\begin{aligned} e^{\mathcal{U}}_*(|c^{\rho,\mathcal{U}}|) &= e^{\mathcal{U}''}_* \circ |i^{\mathcal{U},\mathcal{U}''}|_*(|c^{\rho,\mathcal{U}})|, \text{ by the naturality part of Theorem 8.8} \\ &= e^{\mathcal{U}''}_*(|c^{\rho,\mathcal{U}''}|), \text{ by (9.2)}. \end{aligned}$$

In the same way we have:

$$e^{\mathcal{U}'}_*(|c^{\rho,\mathcal{U}'}|) = e^{\mathcal{U}''}_*(|c^{\rho,\mathcal{U}''}|).$$

So

$$e^{\mathcal{U}}_*(|c^{\rho,\mathcal{U}}|) = e^{\mathcal{U}'}_*(|c^{\rho,\mathcal{U}'}|),$$

and so we are done.

□

We call $c^\rho \in H^{2k}(BG^{top}, \mathbb{R})$ ***the universal Chern-Weil characteristic class associated to*** ρ.

9.1. **Universal cohomological Chern-Weil homomorphism.** Let

$$hcw : \mathcal{I}(G) \to H^*(BG^{top}, \mathbb{R}),$$

be the algebra map sending ρ to c^ρ as above. Then to summarize, we have the following theorem purely about the Milnor classifying space BG^{top}, reformulating Theorem 1.1 of the introduction:

Theorem 9.3. *Let G be a generalized Lie group. The homomorphism hcw satisfies the following property. Let $G \hookrightarrow Z \to Y$ be a smooth principal G-bundle. Let $c^\rho(Z) \in H^{2k}(Y)$ denote the standard Chern-Weil class associated to ρ. Then*

$$f_Z^* hcw(\rho) = c^\rho(Z),$$

where $f_Z : Y \to BG^{top}$ is the classifying map of the underlying topological G-bundle.

Proof. Let $\mathcal{U}_0 \ni Z$ be a G-admissible Grothendieck universe. By Proposition 9.1

$$c^\rho(Z^\Delta) = f^*_{Z^\Delta}(c^{\rho,\mathcal{U}_0}).$$

And by Proposition 7.8, $|c^\rho(Z^\Delta)|_{sm} = c^\rho(Z)$. So we have

$$\begin{aligned} c^\rho(Z) &= |c^\rho(Z^\Delta)|_{sm} \\ &= |f^*_{Z^\Delta}(c^{\rho,\mathcal{U}_0})|_{sm} \\ &= N^*(|f^*_{Z^\Delta} c^{\rho,\mathcal{U}_0}|), \text{ Part 2 of Notation 4.9} \\ &= N^* \circ |f_{Z^\Delta}|^*(|c^{\rho,\mathcal{U}_0}|), \text{ by Lemma 4.10} \\ &= N^* \circ |f_{Z^\Delta}|^* \circ (e^{\mathcal{U}_0})^* c^\rho, \text{ by definition of } c^\rho. \end{aligned}$$

Now, we have a diagram of topological G-bundle maps:

$$\begin{array}{ccccc} |Z^\Delta| & \longrightarrow & Z & \longrightarrow & EG \\ \downarrow & & \downarrow & & \downarrow \\ |Y_\bullet| & \xrightarrow{h} & Y & \xrightarrow{f_Z} & BG^{top}, \end{array}$$

for h as in (4.7). And so $e^{\mathcal{U}} \circ f_{|Z^\Delta|}$ being the classifying map for $|Z^\Delta| \to |Y_\bullet|$, is homotopic to $f_Z \circ h$.

Thus, $e^{\mathcal{U}_0} \circ |f_{Z^\Delta}| \circ N$ is homotopic to f_Z. So that

$$c^\rho(Z) = f_Z^* c^\rho = f_Z^* hcw(\rho),$$

and we are done. □

9.2. Universal dg Chern-Weil homomorphism.

Proof of Theorem 1.4. Let $cw^D : \mathcal{I}(G) \to \Omega^\bullet(BG^{\mathcal{U}})$ be as in the preamble of Section 9. The first part of the theorem readily follows by Lemma 7.4.

Now, let $P \to Y$ be a smooth, $\mathcal{U}$-small G-bundle over a smooth manifold, and set $X = Y_\bullet$. Let $P^\Delta \to X$ denote the induced $\mathcal{U}$-small simplicial G-bundle, and f_{P^Δ} its classifying map. We need to show that cw satisfies the naturality condition:

$$\Theta \circ cw^P \simeq f_{P^\Delta}^* \circ cw.$$

By Proposition 9.1 we have:

$$f_{P^\Delta}^* \circ cw \simeq cw^{P^\Delta}.$$

And by Part 1 of Proposition 7.8 $cw^{P^\Delta} \simeq \Theta \circ cw^P$. And so we are done. □

9.3. **Relation with Whitney-Sullivan de Rham algebra.** Let Y be a topological space and X the simplicial set of singular topological simplices in Y. Set $A(Y) = \Omega^\bullet(X, \mathbb{R})$, then this is a commutative dga.

Note that if X is a Kan complex then we have a homotopy equivalence of simplicial sets $|X|_\bullet \simeq X$ (the natural map $h : X \to |X|_\bullet$ is a weak equivalence, but as both sides are Kan complexes it is then an equivalence.) It follows that for a Kan complex X we have a ***geometric homotopy equivalence*** $h_* : A(|X|) \to \Omega^\bullet(X)$. That is if h^{-1} denotes a homotopy inverse of h, then $h_* \circ h_*^{-1}$ is homotopic to the identity and $h_*^{-1} \circ h_*$ is homotopic to the identity. Furthermore, such a homotopy inverse for h_* is unique up to homotopy.

Proof of Theorem 1.2. Fix any G admissible $\mathcal{U}$. As $BG^{\mathcal{U}}$ is a Kan complex, we have a diagram with each map a homotopy equivalence of simplicial sets:

$$BG^{\mathcal{U}} \xrightarrow{h} |BG^{\mathcal{U}}|_\bullet \xrightarrow{e^{\mathcal{U}_\bullet}} |BG^{Top}|_\bullet.$$

Let $f : A(BG^{Top}) \to \Omega^\bullet(BG^{\mathcal{U}})$ be the dg mapping induced by the composition above. By the discussion of the prior paragraph, and since $e^{\mathcal{U}}$ is also a homotopy equivalence, there is a homotopy inverse f^{-1}, unique up to homotopy. Then set $cw = f^{-1} \circ cw^{\mathcal{U}}$. □

10. Universal Chern-Weil theory for the group of Hamiltonian symplectomorphisms

Let (M, ω) be a possibly non-compact symplectic manifold of dimension $2n$, so that ω is a closed non-degenerate 2-form on M. Let $\mathcal{H} = \mathrm{Ham}(M, \omega)$ denote the group of its compactly generated Hamiltonian symplectomorphisms (as in Section 1.1), and $\mathfrak{h}$ its Lie algebra.

For example, take $M = \mathbb{CP}^{n-1}$ with its Fubini-Study symplectic 2-form ω_{st}. Then the natural action of $PU(n)$ on $\mathbb{CP}^{n-1}$ is by Hamiltonian symplectomorphisms.

In [37] Reznikov constructs multilinear functionals

$$\{r_k\}_{k \geq 1} \subset \mathcal{I}(\mathcal{H}):$$
$$(H_1, \ldots, H_k) \mapsto \int_M H_1 \cdot \ldots \cdot H_k \, \omega^n,$$

upon identifying:

$$\mathfrak{h} = \begin{cases} C_0^\infty(M), & \text{if } M \text{ is compact} \\ C_c^\infty(M), & \text{if } M \text{ is non-compact.} \end{cases}$$

Here $C_0^\infty(M)$ denotes the set of smooth functions H satisfying $\int_M H \, \omega^n = 0$, and $C_c^\infty(M)$ denotes the set of smooth, compactly supported functions. In the case $k = 1$, the associated functional vanishes whenever M is compact.

When M is compact the group $\mathcal{H}$ is a Fréchet Lie group having the homotopy type of a countable CW complex. Otherwise, as mentioned in Section 1.1, it is a generalized Lie group having the homotopy type of a CW complex.

Thus, Theorem 9.3 implies the Corollary 1.5 of the introduction, and in particular we get induced Reznikov cohomology classes

$$c^{r_k} \in H^{2k}(B\mathcal{H}, \mathbb{R}). \tag{10.1}$$

As mentioned, the group $PU(n)$ naturally acts on $\mathbb{CP}^{n-1}$ by Hamiltonian symplectomorphisms. So we have an induced map

$$i : BPU(n) \to B\,\mathrm{Ham}(\mathbb{CP}^{n-1}, \omega_0).$$

Then as one application we prove Theorem 1.7 of the introduction, reformulated as follows:

Theorem 10.1. *[Originally Kedra-McDuff [18]]*

$$i^* : H^k(B\,\mathrm{Ham}(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}) \to H^k(BPU(n), \mathbb{R})$$

is surjective for all $n \geq 2$, $k \geq 0$ and so

$$i_* : H_k(BPU(n), \mathbb{R}) \to H_k(B\,\mathrm{Ham}(\mathbb{CP}^{n-1}, \omega_0), \mathbb{R}),$$

is injective for all $n \geq 2$, $k \geq 0$.

Proof. Let $\mathfrak{g}$ denote the Lie algebra of $PU(n)$, and $\mathfrak{h}$ the Lie algebra of $\mathrm{Ham}(\mathbb{CP}^{n-1}, \omega_0)$. Let $j : \mathfrak{g} \to \mathfrak{h}$ denote the natural Lie algebra map induced by the homomorphism $PU(n) \to \mathrm{Ham}(\mathbb{CP}^{n-1}, \omega_0)$. Reznikov [37] shows that $\{j^* r_k\}_{k>1}$ are the Chern polynomials. In other words, the classes

$$c^{j^* r_k} \in H^{2k}(BPU(n), \mathbb{R}),$$

are the Chern classes $\{c_k\}_{k>1}$, which generate real cohomology of $BPU(n)$, as is well known. But $c^{j^* r_k} = i^* c^{r_k}$, for c^{r_k} as in (10.1), and so the result immediately follows. □

In Kedra-McDuff [18] a proof of the above is given via homotopical techniques. Theirs is a difficult argument, but as they show their technique is also partially applicable to study certain generalized, homotopical analogues of the group $\mathcal{H}$. Our argument is elementary (at least given the general theory), but does not obviously have homotopical ramifications as in [18].

In Savelyev-Shelukhin [41] there are a number of results about induced maps in (twisted) K-theory. These further suggest that the map i above should be a monomorphism in the homotopy category. For a start we may ask:

Question 10.2. *Is the map i above an injection on integral homology?*

For this one may need more advanced techniques like [40].

11. Universal coupling class for Hamiltonian fibrations

Although we use here some language of symplectic geometry no special expertise should be necessary. As the construction here is a partial reformulation of our general constructions, for the special case of $G = \mathcal{H} = \mathrm{Ham}(M, \omega)$, we will not give exhaustive details.

Let (M, ω) and $\mathcal{H}$ be as in the previous section, (keeping in mind our M is not assumed to be compact) and let $2n$ be the dimension of M.

Definition 11.1. *A* **Hamiltonian M-fibration** *is a smooth fiber bundle $M \hookrightarrow P \to X$, with structure group $\mathcal{H}$.*

Each $\mathcal{H}$-connection $\mathcal{A}$ on such P uniquely induces a *coupling 2-form* on P, as originally appearing in [10]. Specifically, this is a closed 2-form $C_{\mathcal{A}}$ on P whose restriction to fibers coincides with ω and which has the following property. Let $\omega_{\mathcal{A}} \in \Omega^2(X)$ denote the 2-form defined by:

$$\omega_{\mathcal{A}}(v, w) = n \int_{P_x} R^{\mathcal{A}}(v, w)\, \omega_x^n,$$

for $v, w \in T_x X$. Here $R^{\mathcal{A}}$ as before is the curvature 2-form of $\mathcal{A}$, so that

$$R^{\mathcal{A}}(v, w) \in \begin{cases} C_0^{\infty}(P_x), & \text{if } M \text{ is compact} \\ C_c^{\infty}(P_x) & \text{if } M \text{ is non-compact.} \end{cases}$$

Note of course that $\omega_{\mathcal{A}} = 0$ when M is compact. The characterizing property of $C_{\mathcal{A}}$ is then:

$$\int_M C_{\mathcal{A}}^{n+1} = \omega_{\mathcal{A}},$$

where the left-hand side is integration along the fiber. [11]

[11] $C_{\mathcal{A}}$ is not generally compactly supported but $C_{\mathcal{A}}^{n+1}$ is, which is a consequence of taking $\mathcal{H}$ to be compactly generated Hamiltonian symplectomorphisms.

It can then be shown that the cohomology class $\mathfrak{c}(P)$ of $C_{\mathcal{A}}$ is uniquely determined by P up to $\mathcal{H}$-bundle isomorphism. This is called the ***coupling class of*** P, and it has important applications in symplectic geometry. See for instance [30] for more details and some applications.

By replacing the category $\mathcal{G}$ with other fiber bundle categories we may define other kinds of simplicial fibrations over a smooth simplicial set. For example, we may replace $\mathcal{G}$ by the category of smooth Hamiltonian M-fibrations, keeping the other axioms in the Definition 5.3 intact. This then gives us the notion of a Hamiltonian simplicial M-bundle over a smooth simplicial set.

Let $\mathcal{U}$ be a $\mathcal{H}$-admissible Grothendieck universe. Let $M^{\mathcal{U},\mathcal{H}}$ denote the Hamiltonian simplicial M-fibration, naturally associated to $E\mathcal{H}^{\mathcal{U}} \to B\mathcal{H}^{\mathcal{U}}$. So that for each k-simplex $\Sigma \in B\mathcal{H}^{\mathcal{U}}$ we have a Hamiltonian M-fibration $M^{\mathcal{U},\mathcal{H}}_{\Sigma} \to \Delta^k$, which is the associated M-bundle to the principal $\mathcal{H}$-bundle $E\mathcal{H}^{\mathcal{U}}_{\Sigma}$.

Fix a (simplicial) $\mathcal{H}$-connection $\mathcal{A}$ on the universal $\mathcal{H}$-bundle $E\mathcal{H}^{\mathcal{U}} \to B\mathcal{H}^{\mathcal{U}}$. This induces a (simplicial) connection with the same name $\mathcal{A}$ on $M^{\mathcal{U},\mathcal{H}}$.

By the discussion above, for each k-simplex $\Sigma \in B\mathcal{H}^{\mathcal{U}}$ we have the associated coupling 2-form $C_{\mathcal{A},\Sigma}$ on the Hamiltonian M-bundle $M^{\mathcal{U},\mathcal{H}}_{\Sigma} \to \Delta^k$. The collection of these 2-forms then readily induces a cohomology class $\mathfrak{c}^{\mathcal{U}}$ on the geometric realization:

$$|M^{\mathcal{U},\mathcal{H}}| = \operatorname{colim}_{\Sigma \in \Delta(B\mathcal{H}^{\mathcal{U}})} M^{\mathcal{U},\mathcal{H}}_{\Sigma}.$$

This is analogous to the construction of the class $|\alpha|$ in Section 4.4.

Now by the proof of Theorem 8.8 we have an $\mathcal{H}$-structure preserving, M-bundle map over the homotopy equivalence $e^{\mathcal{U}}$:

$$g^{\mathcal{U}} : |M^{\mathcal{U},\mathcal{H}}| \to M^{\mathcal{H}},$$

where $M^{\mathcal{H}}$ denotes the universal Hamiltonian M-fibration over $B\mathcal{H}$. And these $g^{\mathcal{U}}$ are natural, so that if $\mathcal{U} \ni \mathcal{U}'$ then

$$[g^{\mathcal{U}'} \circ |\widetilde{i}^{\mathcal{U},\mathcal{U}'}|] = [g^{\mathcal{U}}], \tag{11.1}$$

where $|\widetilde{i}^{\mathcal{U},\mathcal{U}'}| : |M^{\mathcal{U},\mathcal{H}}| \to |M^{\mathcal{U},\mathcal{H}'}|$ is the natural M-bundle map over $i^{\mathcal{U},\mathcal{U}'}$ (as in Theorem 8.8), and where $[\cdot]$ denotes the homotopy class.

Each $g^{\mathcal{U}}$ is a homotopy equivalence, so we may set

$$\mathfrak{c} := g^{\mathcal{U}}_*(\mathfrak{c}^{\mathcal{U}}) \in H^2(M^{\mathcal{H}}).$$

Lemma 11.2. *The class $\mathfrak{c}$ is well-defined, (independent of the choice $\mathcal{U}$).*

The proof is analogous to the proof of Lemma 9.2. Given this definition of the universal coupling class $\mathfrak{c}$, the proof of Theorem 1.6 is analogous to the proof of Theorem 9.3.

A. A_∞ HOMOTOPIES

We prove here the following:

Proposition A.1. *A geometric homotopy of dg maps induces an A_∞ homotopy.*

This is relegated to the appendix for the following reasons. It is somewhat esoteric, and not central to the paper. Moreover, the best reference I have is too general, working with non-unital A_∞ categories and functors rather than unital algebras. In the restricted setting here, the result above should be known. Secondly, the geometric homotopy is a more refined notion and should work better for intended future applications like Cheeger-Simons differential characters. However, A_∞ homotopy notion does have some advantages, as it is purely algebraic and so should be easier to manipulate.

Proof. (Sketch) We omit specifying the coefficient ring $\mathbb{R}$ in what follows. Suppose we are given a geometric homotopy:

$$\widetilde{f} : A \to \Omega^\bullet(X \times I),$$

between dg maps $f_0, f_1 : A \to \Omega^\bullet(X)$.

Define $\widetilde{e}_i : \Omega^\bullet(X) \otimes \Omega^\bullet(I) \to \Omega^\bullet(X)$, $i = 0, 1$, on generators by:

$$\widetilde{e}_i(\omega_0 \otimes \omega_1) = \begin{cases} 0, & \text{if degree } \omega_1 > 0 \\ \omega_0 \cdot \omega_1(i), & \text{if degree } \omega_1 = 0, \end{cases}$$

where $\omega_1(i)$ is evaluation at i.

For e_i as in Definition 7.3, we will factorize:

$$\begin{array}{ccc} \Omega^\bullet(X \times I) & \xrightarrow{e_i} & \Omega^\bullet(X) \\ \downarrow u & \nearrow \widetilde{e}_i & \\ \Omega^\bullet(X) \otimes \Omega^\bullet(I), & & \end{array}$$

where u is a certain A_∞ homotopy inverse to the Kunneth, injective dg quasi-isomorphism $h : \Omega^\bullet(X) \otimes \Omega^\bullet(I) \to \Omega^\bullet(X \times I)$, $h(a \otimes b) = \pi_X^* a \wedge \pi_I^* b$. This will readily imply our claim.

Note first that an A_∞ homotopy inverse exists, see Seidel [43, Corollary 1.14]. But we need u with the specific property above, and we'll construct it following Seidel's argument based on homological perturbation lemma.

To get this u, note first that we have the opposite factorization:

$$\begin{array}{ccc} \Omega^\bullet(X \times I) & \xrightarrow{e_i} & \Omega^\bullet(X), \\ h \uparrow & \nearrow \widetilde{e}_i & \\ \Omega^\bullet(X) \otimes \Omega^\bullet(I), & & \end{array}$$

and the image of h will be denoted by A.

We have a splitting of chain complexes:

$$\Omega^\bullet(X \times I) \simeq A \oplus B, \tag{A.1}$$

where $B = \Omega^\bullet(X \times I)/A$ is acyclic.

Then u is the composition of maps:

$$\Omega^\bullet(X \times I) \longrightarrow A \longrightarrow \Omega^\bullet(X) \otimes \Omega^\bullet(I),$$

where the last map is the natural map (the inverse to h), and the first map as a set map is the projection map with respect to the decomposition (A.1). The latter map can be upgraded to an A_∞ map using the homological perturbation lemma as described in [43, Proof of Corollary 1.14]. □

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