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GLOBAL FUKAYA CATEGORY II: APPLICATIONS

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ABSTRACT. To paraphrase, part I constructs a bundle of A_∞ categories given the input of a Hamiltonian fibration over a smooth manifold. Here we show that this bundle is generally non-trivial by a sample computation. One principal application is differential geometric, and the other is about algebraic K-theory of the integers and the rationals. We find new curvature constraint phenomena for smooth and singular $\mathcal{G}$-connections on principal $\mathcal{G}$-bundles over S^4, where $\mathcal{G}$ is PU(2) or $\mathrm{Ham}(S^2)$. Even for the classical group PU(2) these phenomena are inaccessible to known techniques like the Yang-Mills theory. The above mentioned computation is the geometric component used to show that the categorified algebraic K-theory of the integers and the rationals, defined in [8] following Toën, admits a $\mathbb{Z}$ injection in degree 4. This gives a path from Floer theory to number theory.

1. Introduction

Given a smooth Hamiltonian fibration $M \hookrightarrow P \to X$, with M monotone, and X any smooth manifold, in Part I, that is in [10], we have constructed a functor which may be understood as giving a fibration of A_∞ categories over X, with fiber the Fukaya category of M. This construction is to the Fukaya category as vector bundles are to vector spaces. It will not be necessary to first read Part I to follow this paper, as we will outline most concepts.

More specifically, denote by $\Delta(X)$ the smooth simplex category of X, whose objects are smooth maps $\Sigma : \Delta^d \to X$. See Section 3 for more details. Denote by $wA_\infty Cat_k^{\mathbb{Z}_2}$ the category of strictly unital A_∞ categories over a commutative ring k, with $\mathbb{Z}_2$ graded hom complexes, with morphisms quasi-isomorphisms.

Then on the universal level, in Part I we constructed a functor:

$$F_k : \Delta(B\,\mathrm{Ham}(M,\omega)) \to wA_\infty Cat_k^{\mathbb{Z}_2}, \tag{1.1}$$

having some additional structure, which facilitates some "homotopy theory". Here we show, for $k = \mathbb{Z}, \mathbb{Q}$, that this functor is (generally) "homotopically" non-trivial, and give some applications for this fact. Homotopic non-triviality is more specifically a condition on the concordance type of F_k, see Definition 4.6.

As explained in Part I, the above homotopy non-triviality may be understood to say that there is a non-trivial "homotopy coherent" action of $\mathrm{Ham}(M,\omega)$ on $\mathrm{Fuk}(M,\omega)$. Existence of such an action in a somewhat weaker form (on the level E_2 algebras rather than space level) has been independently conjectured by Teleman, ICM 2014, post-factum of appearance of Part I.

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1.1. **Method of Cartesian and Kan fibrations.** Given our Hamiltonian bundle $M \hookrightarrow P \to X$, analogously to (1.1), we get a functor:

$$F_{P,k} : \Delta(X) \to wA_\infty Cat_k^{\mathbb{Z}_2}.$$

Let $\mathrm{NFuk}(M,\omega)$, denote the Faonte-Lurie A_∞ nerve of $\mathrm{Fuk}(M,\omega)$, (see Appendix A.4 of Part I.) Denote by $X_\bullet$ the smooth singular set, that is the simplicial set with n-simplices smooth maps $\Delta^n \to X$. As we will recall in Section 4 $F_{P,\mathbb{Q}}$ induces a simplicial fibration:

$$\mathrm{NFuk}(M,\omega) \hookrightarrow \mathrm{Fuk}_\infty(P) \xrightarrow{p_\bullet} X_\bullet.$$

This simplicial fibration is furthermore a Cartesian fibration (in this case equivalently categorical fibration).

We show that for P a non-trivial Hamiltonian S^2 fibration over S^4, the maximal Kan sub-fibration of $\mathrm{Fuk}_\infty(P)$ is non-trivial. In particular, $\mathrm{Fuk}_\infty(P)$ is non-trivial as a Cartesian fibration. The latter readily implies:

Theorem 1.1. *For $M = S^2$ the functors $F_\mathbb{Z}, F_\mathbb{Q}$ are not null-concordant. Furthermore, $F_{P,\mathbb{Q}}$ is not concordant to $F_{P',\mathbb{Q}}$ if $P \not\simeq P'$ are Hamiltonian S^2 bundles over S^4.*

The proof of the theorem above makes use of a certain computation from [9], using the notion of quantum Maslov classes. The following is a partial corollary, the proof of which will only be outlined here, due to some algebraic dependencies that are beyond the scope here.

Theorem 1.2. *There is a natural injection*

$$\mathbb{Z} \to K_4^{Cat,\mathbb{Z}_2}(k),$$

where $k = \mathbb{Z}, \mathbb{Q}$ and the right hand side is the categorified algebraic K-theory group, introduced in [8], following ideas of Toën [12].

The groups $K_n^{Cat,\mathbb{Z}_2}(\mathbb{Z})$ encode some arithmetic information, (at the moment mysterious) that should be a strict refinement of the basic algebraic K-theory groups $K_n(\mathbb{Z})$. The latter groups are only partially computed, but at least we do have some idea of the arithmetic information contained therein. It is a theorem of Rognes [7] that $K_4(\mathbb{Z}) = 0$. As explained in [8, Example 1], this together with the result above implies that the natural trace maps from $K^{Cat,\mathbb{Z}_2}(\mathbb{Z})$ to $K(\mathbb{Z})$ have a kernel. So we get a connection from Floer theory to algebraic K-theory and arithmetic.

1.2. **An application to curvature bounds of singular and smooth connections.**

1.2.1. *A non-metric measure of curvature.* The geometry of our calculation will naturally tie in with some theory of singular connections, which we now outline. Let G as above be a Frechet Lie group, we denote by $\mathrm{lie}\, G$ its Lie algebra. Let

$$\mathfrak{n} : \mathrm{lie}\, G \to \mathbb{R}$$

be an Ad invariant Finsler norm. For a principal G-bundle P over a Riemann surface (S, j), and given a G connection $\mathcal{A}$ on P define a 2-form $\alpha_\mathcal{A}$ on S by:

$$\alpha_\mathcal{A}(v, jv) = \mathfrak{n}(R_\mathcal{A}(v, jv)),$$

where $R_{\mathcal{A}}$ is the curvature 2-form of $\mathcal{A}$. More specifically, the latter form has the properties:

$$R_{\mathcal{A}}(v, w) \in \operatorname{lie} \operatorname{Aut} P_z,$$

for $z \in S$, $v, w \in T_z S$, P_z the fiber of P over z, $\operatorname{Aut} P_z \simeq G$ the group of G-bundle automorphisms of P_z, and where $\simeq$ means non-canonical group isomorphism.

Define

$$\text{energy}_{\mathfrak{n}}(\mathcal{A}) := \int_S \alpha_{\mathcal{A}}. \tag{1.2}$$

In the case $\mathcal{A}$ is singular with singular set C, $\alpha_{\mathcal{A}}$ is defined on $S - C$, so we define

$$\text{energy}_{\mathfrak{n}}(\mathcal{A}) := \int_{S-C} \alpha_{\mathcal{A}},$$

with the right-hand side now being an extended integral. This energy is a non-metric measurement meaning that no Riemannian metric on S is needed.

It is possible to extend the functional above to a functional on the space $\mathcal{C}$ of G-connections on principal G bundles $P \to \Delta^n$. It may seem that Δ^n has no connection to Riemann surfaces, but in fact there is an intriguing such connection. Let $\mathcal{S}_d$ denote the universal curve over $\overline{\mathcal{R}}_d$ - the moduli space of complex structures on the disk with $d+1$ punctures on the boundary. And let $\mathcal{S}_d^\circ$ denote $\mathcal{S}_d$, with nodal points of the fibers removed. Then it is shown in Part I that there are certain axiomatized systems of maps:

$$u : \mathcal{S}_d^\circ \to \Delta^n, \quad \text{with } d, n \text{ varying}.$$

Such a system is uniquely determined up to suitable homotopy, and is referred to as $\mathcal{U}$.

There is then a natural functional:

$$\text{energy}_{\mathcal{U}} : \mathcal{C} \to \mathbb{R}_{\geq 0}, \tag{1.3}$$

defined with respect to a choice of $\mathcal{U}$, see Section 11. When $n = 2$ it is just the energy functional as previously defined.

1.2.2. *Curvature bounds.* The following basic result can be formally understood as a corollary of Theorem 1.7. But it is more elementary to see it as a corollary of Theorem 11.4.

Corollary 1.3 (Of Theorem 1.7 and of Theorem 11.4)**.** *Let P be a non-trivial* $\mathrm{PU}(2)$ *(or* $\mathrm{Ham}(S^2)$*) bundle $P \to S^4$, let $D^4_\pm$ be as above and let $\mathcal{A}$ be a smooth $PU(2)$ or* $\mathrm{Ham}(S^2, \omega)$ *connection on P. For any $\mathcal{U}$:*

$$\text{energy}_{\mathcal{U}}(\mathcal{A})\,|_{D^4_-} < \frac{1}{2} \implies \text{energy}_{\mathcal{U}}(\mathcal{A})\,|_{D^4_+} \geq \frac{1}{2}.$$

Remark 1.4. *The proof of even the above corollary traverses the entirety of the theory here. As this is a very elementary result we may hope for a simpler argument. In the case of $PU(2)$, one idea might be to replace Floer theory, used here, by the technically simpler mathematical Yang-Mills theory over surfaces* [1]*. If we want to mimic the argument presented in this paper, then we should first extend Yang-Mills theory to work with G-bundles over surfaces with corners and holonomy constraints over boundary. This should be possible, but beyond this things are unclear, since,* *as we also use certain abstract algebraic topology to glue various analytic data, and it is not clear how this would work for Yang-Mills theory.*

1.2.3. *Abstract resolutions of singular connections.* We can use the computation of Theorem 1.1 to obtain lower bounds for the curvature of certain types of singular connections.

Definition 1.5. *Let $G \hookrightarrow P \to X$ be a principal G bundle, where G is a Frechet Lie group. A* **singular G-connection** *on P is a closed subset $C \subset X$, and a smooth Ehresmann G-connection $\mathcal{A}$ on $P|_{X-C}$.*

The above definition is basic, as one often puts additional conditions, see for instance [2], [11].

Avoiding generality, suppose that $\mathcal{A}$ is a singular G-connection on a principal G-bundle $P \to S^n$, with a single singularity x_0. We will show that it is possible to control the curvature of the singular connection $\mathcal{A}$ if we impose a certain structure on the singularity of $\mathcal{A}$. The simplest way to do this is to ask for existence of a certain kind of abstract resolution.

First, a simplicial G-connection $\mathcal{A}$ on P, as defined in Section 11.1, is a functorial assignment

$$\Sigma \mapsto \mathcal{A}_\Sigma,$$

where $\mathcal{A}_\Sigma$ is a smooth G-connection on $\Sigma^* P$, for each smooth

$$\Sigma : \Delta^n \to S^n.$$

Definition 1.6. *For $\mathcal{A}, P$ as above a* **simplicial resolution** *of $\mathcal{A}$ is a simplicial G connection $\mathcal{A}^{res}$ on P, with the following property. Let $\Sigma_0 : \Delta^n \to S^n$ represent the generator of $\pi_n(S^n, x_0)$ (cf. Appendix A), then*

$$(\Sigma_0|_{interior\, \Delta^n})^* \mathcal{A} = inc^* \mathcal{A}^{res}_{\Sigma_0},$$

where $inc : interior\, \Delta^n \to \Delta^n$ is the inclusion map.

In the following theorem $G = PU(2), n = 4$ and the norm $\mathfrak{n}$ on lie $PU(2)$ will be taken to be the operator norm, normalized so that the Finsler length of the shortest one parameter subgroup from id to $-id$ is $\frac{1}{2}$. We will omit $\mathfrak{n}$ in notation. We also impose an additional constraint on $\mathcal{A}^{res}$, so that the curvature at "∞" is bounded by a threshold, which means the following. Let

$$\Sigma_\infty : \Delta^4 \to x_0$$

be the constant map. Suppose that:

$$\text{energy}_{\mathcal{U}}\, \mathcal{A}^{res}_{\Sigma_\infty} < 1/2,$$

and suppose for simplicity that $\mathcal{A}^{res}_{\Sigma_\infty}$ is trivial along the edges of Δ^4, later on this condition is relaxed, see Proposition 11.4. (This condition can be completely removed but at the cost of significant additional complexity.)

We say in this case that $\mathcal{A}^{res}$ is a ***sub-quantum resolution***. The following is proved in Section 11.

Theorem 1.7. *Let $P \to S^4$ be a non-trivial principal $PU(2)$ bundle. Let $\mathcal{A}$ be a singular $PU(2)$-connection on P with a single singularity at x_0. Then for any sub-quantum resolution $\mathcal{A}^{res}$ of $\mathcal{A}$ and for any $\mathcal{U}$ as above*

$$\text{energy}_{\mathcal{U}}(\mathcal{A}^{res}_{\Sigma_0}) \geq 1/2.$$

The theorem has certain extensions to Hamiltonian singular connections $\mathcal{A}$, understanding P as a principal $\text{Ham}(S^2)$ bundle, see Section 11.

Example 1.8. *Let P be as above, and $\mathcal{A}'$ be an ordinary smooth $PU(2)$ connection on P. Express S^4 as a union of sub-balls $D^4_\pm \subset S^4$, intersecting only in the boundary. Suppose that we have the property that $\text{energy}_{\mathcal{U}}(\mathcal{A}')|_{D^4_-} < \frac{1}{2}$. Let $\mathcal{A}$ be the singular connection on P obtained as the push-forward of $\mathcal{A}'$ by the bundle map $\widetilde{q} : P \to P$ over the singular smooth map $q : S^4 \to S^4$ taking D^4_- to a single point $\infty \in S^4$, with $q|_{interior\, D^4_-}$ an immersion. Then $\mathcal{A}$ has a sub-quantum resolution $\mathcal{A}^{res}$ by construction. In this case, the theorem above simply yields that* $\text{energy}_{\mathcal{U}}(\mathcal{A}'_{D^4_+}) \geq 1/2$.

Remark 1.9. *There are possible physical interpretations for singular connections, as appearing in the context here. A $PU(2)$ connection $\mathcal{A}$ on P in physical terms represents a Yang-Mills field on the space-time S^4. When the space-time has a black hole singularity, the fields solving the Einstein-Yang-Mills equations (mathematically connections as above) likewise develop singularities. There is a wealth of physics literature on this subject, here is one sample reference [5]. As quantum gravity is often related to simplicial ideas, it is not inconceivable that the mathematical sub-quantum resolution condition above also has a (quantum gravity theoretic) physical interpretation.*

At this point the reader may be curious why Theorem 1.1 has something to do with Theorem 1.7. We cannot give the full story, but the idea is that the Cartesian fibration $Fuk_\infty(P)$ only sees the principal bundle P (and its curvature) by the behavior of certain holomorphic curves. When one has the sub-quantum condition on the curvature of $\mathcal{A}^{res}_{\Sigma_\infty}$, certain holomorphic curves are ruled out so that from the view point of $Fuk_\infty(P)$, $\mathcal{A}^{res}_{\Sigma_\infty}$ is the trivial connection, (its curvature is undetectable) but $Fuk_\infty(P)$ is non-trivial as a fibration so that the aforementioned holomorphic curves and consequently curvature must appear elsewhere.

1.3. **First quantum obstruction and smooth invariants.** We present here a construction of an integer valued smooth invariant which is based on our theory. This is probably just the beginning of the story for invariants of smooth manifolds based on Floer-Fukaya theory.

First we discuss a more general setup. Let $M \hookrightarrow P \xrightarrow{p} X$ be a Hamiltonian M-bundle, as previously. Let

$$\text{Fuk}_\infty(P) \xrightarrow{p_\bullet} X_\bullet$$

be the associated Cartesian fibration, and let

$$K(P) \to X_\bullet$$

be its maximal Kan sub-fibration as in Lemma 4.2. Then $|K(P)| \to X$ is a Serre fibration, where $|K(P)|$ is the geometric realization.

Define

$$\text{q-obs}(P) \in \mathbb{N} \sqcup \{\infty\},$$

to be the degree of the first obstruction to a section of $|K(P)|$. That is q-obs(P) is the smallest integer n such that there is no section of $|K(P)|$ over the n skeleton of X, with respect to some chosen CW structure. This is independent of the choice of the CW structure, as any pair of CW structures on X are filtered (using cellular filtration) homotopy equivalent up to a wedge sum with some collection of D^n, $n \in \mathbb{N}$ (with its canonical CW structure), see Faria [4, Theorem 2.4].

When no such n exists we set q-obs$(P) = \infty$.

Theorem 1.10. *Let $S^2 \hookrightarrow P \to S^4$ be a non-trivial Hamiltonian fibration then:*

$$\text{q-obs}(P) = 4.$$

In fact we prove that the obstruction class in

$$H^4(S^4, \pi_3(\text{NFuk}(S^2))$$

is non-trivial.

1.3.1. *First quantum obstruction as a manifold invariant.* Let X be a smooth manifold, and let $P(X)$ denote the fiber-wise projectivization of $TX \otimes \mathbb{C}$. We then define

$$\text{q-obs}(X) := \text{q-obs}(P(X)) \in \mathbb{N} \sqcup \{\infty\},$$

which is then an invariant of the smooth manifold X. It should be noted that this "first quantum obstruction" invariant is only sensitive to the tangent bundle, whereas for example Donaldson invariants can see finer aspects of the smooth structure. In fact the "quantum Novikov conjecture" of Part I would immediately imply that the first quantum obstruction is only a topological invariant of X, but in that case it is very likely a new topological invariant (not expressible in terms like Pontryagin numbers).

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Contents

3. Preliminaries

We quickly recall some notation from Part I and some other basics for convenience. We denote by Δ the simplex category:

- The set of objects of Δ is $\mathbb{N}$.
- $\hom_\Delta(n, m)$ is the set of non-decreasing maps $[n] \to [m]$, where $[n] = \{0, 1, \ldots, n\}$, with its natural order.

A simplicial set X is a functor

$$X : \Delta^{op} \to Set.$$

The set $X(n)$ is called the set of n-simplices of X. $\Delta^d_\bullet$ will denote a particular simplicial set: the standard representable d-simplex, with

$$\Delta^d_\bullet(n) = hom_\Delta(n, d).$$

We use notation Δ^n to denote the standard topological n-simplex.

Definition 3.1. *For Y smooth manifold, $\Delta(Y)$ will denote the* **smooth simplex category of Y**. *This is the category s.t.:*

- *The set of objects* $\operatorname{obj}\Delta(Y)$ *is the set of smooth maps:*
$$\Sigma : \Delta^d \to X, \quad d \geq 0.$$
- *Morphisms $f : \Sigma_1 \to \Sigma_2$ are commutative diagrams in $s-Set$:*

$$\begin{array}{ccc} \Delta^d & \longrightarrow & \Delta^n, \\ & \searrow^{\Sigma_1} & \downarrow \Sigma_2 \\ & & X. \end{array} \tag{3.1}$$

  where $\Delta^n \to \Delta^m$ is a simplicial map, that is an affine map taking vertices to vertices, preserving the order.

When Y is a smooth manifold $Y_\bullet$ will denote the smooth singular set of X. That is $X_\bullet(n)$ is the set of smooth maps $\Delta^n \to X$.

If $p : X \to Y$ is a map of spaces, $p_\bullet : X_\bullet \to Y_\bullet$ will mean the induced simplicial map. We will denote abstract Kan complexes or quasi-categories by calligraphic letters e.g. $\mathcal{X}, \mathcal{Y}$.

4. Outline

In what follows, when we say Part I we shall mean [10]. We will mostly follow the notation and setup of Part I, as well as some setup of [9]. The reader may review the basics of simplicial sets, in Section 3 of Part I. For a more detailed introduction, which also includes some theory of quasi-categories, we recommend Riehl [6]. Here are some specific summary points. Let us briefly review what we do in Part I. Let $M \hookrightarrow P \xrightarrow{p} X$ be a Hamiltonian fibration (with monotone compact fibers).

As in Part I, an auxiliary perturbation data $\mathcal{D}$ for P, (in particular) involves:

- A choice of a natural system $\mathcal{U}$, see Section 8, consisting of certain maps
$$u : \mathcal{S}_d^\circ \to \Delta^n, \text{with } d, n \text{ varying},$$
  as already discussed in Section 1.2.1 of the Introduction.
- Choices of certain Hamiltonian connections, on Hamiltonian bundles associated to the maps u above. We partially review this in Sections 7, 8.

Let $A_\infty Cat_k^{\mathbb{Z}_2}$ denote the category of cohomologically unital A_∞ categories over a commutative ring k, with $\mathbb{Z}_2$ graded hom complexes, with morphisms $\mathbb{Z}_2$ graded A_∞ functors. Given $\mathcal{D}$ as above, in Part I we constructed a functor
$$F_{P,k} : \Delta(X) \to A_\infty Cat_k^{\mathbb{Z}_2}.$$
This functor has certain additional properties.

The functor $F_{P,k}$ takes all morphisms to quasi-isomorphisms. Moreover, there is an algebraically induced functor
$$F_{P,k}^{unit} : \Delta(X) \to wA_\infty Cat_k^{\mathbb{Z}_2},$$

where $wA_\infty Cat^{\mathbb{Z}_2}_{\mathbb{Q}} \subset A_\infty Cat^{\mathbb{Z}_2}$ denotes the sub-category consisting of strictly unital A_∞ categories, with morphisms unital quasi-isomorphisms. $F^{unit}_{P,k}$ is constructed by taking unital replacements.

Notation 4.1. *In what follows we rename $F_{P,k}$ by F^{raw}_P (raw as in having the raw data of holomorphic curves) and $F^{unit}_{P,k}$ by F_P, with k implicit. Then the notation matches the one in the Introduction.*

Let N be the A_∞ nerve functor of Faonte-Lurie as mentioned in the Introduction. Taking $k = \mathbb{Q}$ we defined in Part I:

$$\mathrm{Fuk}_\infty(P) = \mathrm{colim}_{\Delta(X)} N \circ F_P. \tag{4.1}$$

The latter is shown in Part I to be an ∞-category, and moreover it has the structure of a Cartesian fibration:

$$\mathrm{NFuk}(M, \omega) \hookrightarrow \mathrm{Fuk}_\infty(P) \xrightarrow{p_\bullet} X_\bullet.$$

The equivalence class of the latter under concordance, see Definition 4.4, is independent of all choices. Note that this is also a property of F_P.

In general, given a functor

$$F : \Delta(X) \to wA_\infty Cat^{\mathbb{Z}_2}_{\mathbb{Q}},$$

$$\mathrm{hocolim}_{\Delta(X)} N \circ F, \tag{4.2}$$

is a Cartesian fibration over $X_\bullet$. Or more properly stated, there is a representation of this homotopy colimit that is naturally a Cartesian fibration over the classical nerve $N(\Delta(X))$. The latter nerve is weakly equivalent to $X_\bullet$. To get such a representation we can use the "Grothendieck construction" see Lurie [3, Definition A.3.5.11].

Now for F_P as above

$$\mathrm{hocolim}_{\Delta(X)} N \circ F_P \simeq Fuk_\infty(P)$$

as Cartesian fibrations. This is again a property of F_P, (it is a fibrancy type condition). The latter assertion is not proved in Part I, but readily follows the setup of Part I and basic theory of colimits of simplicial sets.

4.1. **From a Cartesian fibration to a Kan fibration.** We will extract from $Fuk_\infty(P)$ a Kan fibration and work with that, since then we can just use standard tools of topology.

To this end we have the following elementary lemma.

Lemma 4.2. *Suppose we have a Cartesian fibration $p : \mathcal{Y} \to \mathcal{X}$, where $\mathcal{X}$ is a Kan complex. Let $K(\mathcal{Y})$ denote the maximal Kan sub-complex of $\mathcal{Y}$ then $p : K(\mathcal{Y}) \to \mathcal{X}$ is a Kan fibration.*

The proof is given in Appendix A. In particular by the above lemma

$$K(P) := K(\mathrm{Fuk}_\infty(P)) \xrightarrow{p_\bullet} X_\bullet$$

is a Kan fibration.

Notation 4.3. *In what follows $p_\bullet$ will refer to this projection unless specified otherwise.*

Definition 4.4. *We say that a Kan fibration, or a Cartesian fibration $\mathcal{P}$ over a Kan complex $\mathcal{X}$ is* **non-trivial** *if it is not null-concordant. Here $\mathcal{P}$ is* **null-concordant** *means that there is a Kan respectively Cartesian fibration*

$$\mathcal{Y} \to \mathcal{X} \times \Delta^1_\bullet,$$

whose pull-back by $i_0 : \mathcal{X} \to \mathcal{X} \times \Delta^1_\bullet$ is trivial and by $i_1 : \mathcal{X} \to \mathcal{X} \times \Delta^1_\bullet$ is $\mathcal{P}$. Here the two maps i_0, i_1 correspond to the two vertex inclusions $\Delta^0_\bullet \to \Delta^1_\bullet$.

Theorem 4.5. *Suppose that $p : P \to S^4$ is a non-trivial Hamiltonian S^2 fibration then $p_\bullet : K(P) \to S^4_\bullet$ does not admit a section. In particular $K(P)$ is a non-trivial Kan fibration over $S^4_\bullet$ and so $Fuk_\infty(P)$ is a non-trivial Cartesian fibration over $S^4_\bullet$.*

This is the main technical result of the paper. Although in a sense we just are just deducing existence of a certain holomorphic curve, for this deduction we need a global compatibility condition involving multiple moduli spaces, involved in multiple local datum's of Fukaya categories, so that this computation will not be straightforward.

Definition 4.6. *Let $I = [0, 1]$. We say that two functors*

$$F_0, F_1 : \Delta(X) \to wA_\infty Cat_k^{\mathbb{Z}_2}$$

are **concordant** *if there is a functor*

$$\widetilde{F} : \Delta(X \times I) \to wA_\infty Cat_k^{\mathbb{Z}_2}$$

s.t. $\widetilde{F}|_{\Delta(Y \times \{0\})} = F_0$ and $\widetilde{F}|_{\Delta(Y \times \{1\})} = F_1$.

Proof of Theorem 1.1. Suppose otherwise that we have a concordance $\widetilde{F}$ of $F_{\mathbb{Z}}$ to a constant functor $Const$. Then tensoring (all the relevant chain complexes) with $\mathbb{Q}$ we get a null-concordance of $F_{\mathbb{Q}}$. Applying the construction (4.2) we get a null-concordance of the Cartesian fibration $Fuk_\infty(P)$, for P as in Theorem 4.5. But this is a contradiction to the latter theorem.

The second part of the theorem is deduced as follows. Suppose by contradiction that $P \not\simeq P'$ and F_P is concordant to $F_{P'}$ so that

$$Fuk_\infty(P) \sim Fuk_\infty(P'), \tag{4.3}$$

where $\sim$ denotes the concordance relation.

Now $P = P_g$ and $P' = P_{g'}$, where the right hand side is as in Section 9.1. That is g, g' are the corresponding clutching maps in $\pi_3 \operatorname{Ham}(S^2)$, and $g \neq g'$.

Let $\#$ denote the natural bundle connect sum operation. We have:

$$\begin{aligned} Fuk_\infty(P_{g^{-1} \cdot g'}) &\simeq Fuk_\infty(P_{g^{-1}}) \# Fuk_\infty(P_{g'}) \text{ by basic topology and definitions} \\ &\sim Fuk_\infty(P_{g^{-1}}) \# Fuk_\infty(P_g) \text{ by (4.3)} \\ &\simeq Fuk_\infty(P_{g^{-1} \cdot g}). \end{aligned}$$

So we get that $Fuk_\infty(P_{g^{-1} \cdot g'})$ is null-concordant. But $P_{g^{-1} \cdot g'}$ is by assumption a non-trivial bundle, so that we get a contradiction to the first part of the theorem. □

Proof of Theorem 1.2. We can only sketch this, as there is much prerequisite algebra that is beyond the scope here. Recall that a finite dimensional complex vector bundle over S^n, induces a non-trivial class in $\pi_n(KU \simeq BU \times \mathbb{Z})$ if it has a non-trivial Chern class. The proof of the corollary is analogous.

Note that $\pi_4(\text{BHam}(S^2, \omega)) \simeq \pi_3(SO(3)) = \mathbb{Z}$, by Smale's theorem. By [8, Corollary 6.2] we get a homomorphism:

$$\mathbb{Z} \simeq \pi_4(\text{BHam}(S^2, \omega)) \to K_4^{Cat, \mathbb{Z}_2}(k).$$

The proof of Theorem 1.1 is implicitly by a computing a certain characteristic class associated to a functor of type $F_{P,\mathbb{Q}}$. [1] And these characteristic classes determine P up to isomorphism. In a future work we will make these characteristic classes explicit.

Given this, one proceeds as in the classical case of complex vector bundles, as recalled above. □

The proof of Theorem 4.5 will be aided by constructing suitable perturbation data, and will be split into a number of sections.

5. Qualitative description of the perturbation data

We are now going to discuss the perturbation data needed for the construction of the functor $F_P : \Delta(S^4) \to wA_\infty Cat_{\mathbb{Q}}^{\mathbb{Z}_2}$, where P from now on is as in Theorem 4.5.

Let $\text{Fuk}(S^2)$ denote the $\mathbb{Z}_2$ graded A_∞ category over $\mathbb{Q}$, with objects oriented spin Lagrangian submanifolds Hamiltonian isotopic to the equator. Our particular construction of $\text{Fuk}(M, \omega)$ is presented in Part I. To briefly outline, as part of some perturbation data $\mathcal{D}$, for each $L, L' \in \text{obj}\,\text{Fuk}(M, \omega)$ we have Hamiltonian connections $\mathcal{A}(L, L')$ on the trivial fibration $[0,1] \times M \to [0,1]$. The homomorphism complex

$$\hom_{\text{Fuk}(M,\omega)}(L, L')$$

is the complex

$$CF(L, L', \mathcal{A}(L, L'))$$

defined as in Section 6.1 of Part I. To paraphrase, the latter is the Floer chain complex generated over $\mathbb{Q}$ by $\mathcal{A}(L, L')$-flat sections of $[0,1] \times M$, with boundary on $L \subset \{0\} \times M, L' \subset \{1\} \times M$. The homology of $CF(L, L', \mathcal{A}(L, L'))$ will be denoted by $FH(L, L')$, understood as a $\mathbb{Z}_2$ graded abelian group.

The data $\mathcal{D}$ is generally associated to a Hamiltonian fibration. As a symplectic manifold is a Hamiltonian fibration over a point, we write $\mathcal{D}_{pt}$ for this restricted data, needed for construction of $\text{Fuk}(S^2)$ as outlined above.

[1] This is something like the quantum Maslov class also appearing here but defined algebraically in the context of a general functor $F : \Delta(X) \to wA_\infty Cat_k^{\mathbb{Z}_2}$.

Denote by $\mathrm{Fuk}^{\mathrm{eq}}(S^2) \subset \mathrm{Fuk}(S^2)$ the full sub-category obtained by restricting our objects to be oriented standard equators in S^2. We take our perturbation data $\mathcal{D}_{pt}$ so that the following is satisfied.

- All the connections $\mathcal{A}(L, L')$ as above are $PU(2)$-connections.
- For L intersecting L' transversally, the $PU(2)$ connection $\mathcal{A}(L, L')$ is the trivial flat connection.
- For $L = L'$ the corresponding connection is generated by an autonomous Hamiltonian.

The associated cohomological Donaldson-Fukaya category $DF(S^2)$ is equivalent as a linear category over $\mathbb{Q}$ to $FH(L_0, L_0)$ (considered as a linear category with one object) for $L_0 \in \mathrm{Fuk}(S^2)$.

It is easily verified that a morphism (1-edge) f is an isomorphism in the nerve $\mathrm{NFuk}(S^2)$, if and only if it corresponds, under the nerve construction N, to a morphism in $\mathrm{Fuk}(S^2)$ that induces an isomorphism in $DF(S^2)$. Such a morphism will be called a *c-isomorphism*.

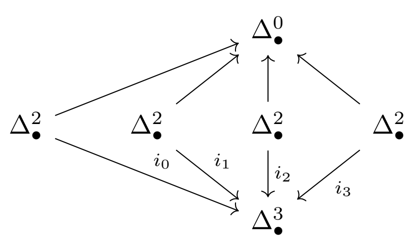
Consequently the maximal Kan subcomplex $K(S^2)$ of $\mathrm{NFuk}(S^2)$ is characterized as the maximal subcomplex with 1-simplices the images by N of c-isomorphisms in $Fuk(S^2)$.

5.1. **Extending $\mathcal{D}_{pt}$ to higher dimensional simplices.**

Terminology 5.1. *A bit of possibly non-standard terminology: we say that A is a* model *for B in some category, with weak equivalences, if there is a morphism $mod : A \to B$ which is a weak-equivalence. The map mod will be called a* modelling map. *In our context the modeling map mod always turns out to be a monomorphism.*

Let us model $D^4_\bullet$ and $S^3_\bullet$ as follows. Take the standard representable 3-simplex $\Delta^3_\bullet$, and the standard representable 0-simplex $\Delta^0_\bullet$. Then collapse all faces of $\Delta^3_\bullet$ to a point, that is take the colimit of the following diagram:

(5.1)



Here i_j are the inclusion maps of the non-degenerate 2-faces. This gives a simplicial set $S^{3,mod}_\bullet$ modeling the simplicial set $S^3_\bullet$, in other words there is a natural a weak-equivalence

$$S^{3,mod}_\bullet \to S^3_\bullet.$$

Now take the cone on $S^{3,mod}_\bullet$, denoted by $C(S^{3,mod}_\bullet)$, and collapse the one non-degenerate 1-edge. The resulting simplicial set $D^{4,mod}_\bullet$ is our model for $D^4_\bullet$, it may be identified with a subcomplex of $D^4_\bullet$ so that the inclusion map $mod : D^{4,mod}_\bullet \to D^4_\bullet$ induces a weak homotopy equivalence of pairs

$$(D^{4,mod}_\bullet, S^{3,mod}_\bullet) \to (D^4_\bullet, S^3_\bullet). \tag{5.2}$$

We set $b_0 \in D^4_\bullet$ to be the vertex which is the image by *mod* of the unique 0-vertex in $D^{4,mod}_\bullet$.

Suppose we have a commutative diagram:

$$\begin{array}{ccccc} D^4 & \xrightarrow{h_+} & S^4 & \xleftarrow{h_-} & D^4 \\ & \nwarrow^{i} & & \nearrow_{i} & \\ & & S^3 & & \end{array}$$

where $i : S^3 \to D^4$ is the natural boundary inclusion, and s.t. the following is satisfied.

- $h_\pm : D^4 \to S^4$ are smooth, and their images cover S^4.
- $$h_+(D^4) \cap h_-(D^4)$$
  is contained in the image E of
  $$h_\pm \circ i : S^3 \to S^4.$$
- $h_\pm$ takes b_0 to x_0.

For example, we may just let h_- represent the generator of $\pi_4(S^4, x_0)$ and h_+ to be the constant map to x_0. We call such a pair $h_\pm$ a *complementary pair*.

We set
$$D_\pm := h_\pm(D^{4,mod}_\bullet) \subset S^4_\bullet,$$
and we set $\Sigma_\pm \in S^4_\bullet$ to be the image by $h_\pm$ of the sole non-degenerate 4-simplex of $D^{4,mod}_\bullet$. Also set
$$\partial D_\pm := h_\pm(\partial D^{4,mod}_\bullet),$$
where $\partial D^{4,mod}_\bullet$ is the image of the natural inclusion $S^{3,mod}_\bullet \to D^{4,mod}_\bullet$.

Fix a Hamiltonian frame for the fiber P_{x_0} of P over x_0, in other words a Hamiltonian bundle diffeomorphism

$$\begin{array}{ccc} S^2 & \longrightarrow & P \\ \downarrow & & \downarrow{\scriptstyle p} \\ pt & \xrightarrow{x_0} & S^4. \end{array} .$$

In particular, this allows us to identify $\mathrm{Fuk}(S^2)$ with $F^{raw}(x_0)$, using the analytic perturbation data $\mathcal{D}_{pt}$ for both. Denote by $x_{0,\bullet}$ the image of the map

$$\Delta^0_\bullet \to S^{4,mod}_\bullet,$$

induced by the inclusion of the 0-simplex x_0.

We continue with the description of the data $\mathcal{D} = \mathcal{D}(P)$. This must associate certain data $\mathcal{D}_\Sigma$ for each singular simplex $\Sigma : \Delta^n \to S^4$. Now in Section 8 of Part I, given the data $\mathcal{D}_\Sigma$ for a non-degenerate simplex Σ, we assigned extended perturbation data $\mathcal{D}_{\widetilde{\Sigma}}$ for all degeneracies $\widetilde{\Sigma}$ of this simplex. So by that discussion, our chosen data $\mathcal{D}_{pt}$ induces perturbation data for all degeneracies of x_0, that is for all simplices of $x_{0,\bullet}$, this data will again be denoted by $\mathcal{D}_{pt}$, for simplicity.

Fix an object

$$L_0 \in \mathrm{Fuk}^{\mathrm{eq}}(S^2) \subset F^{raw}(x_0). \tag{5.3}$$

Denote by $\gamma \in \hom_{F^{raw}(x_0)}(L_0, L_0)$ the generator of $FH_1(L_0, L_0)$, i.e. the fundamental chain, so that it corresponds to the identity in $DF(L_0, L_0)$. This γ is uniquely determined by our conditions and corresponds to a single geometric section. Denote by L_0^i the image of L_0 by the embedding

$$F^{raw}(x_0) \to F^{raw}(\Sigma_+),$$

corresponding to the i'th vertex inclusion into Δ^4, $i = 0, \ldots, 4$.

Let m_i be the edge between $i-1, i$ vertices and set

$$\overline{m}_i := \Sigma_+ \circ m_i.$$

Let Σ_i^0 denote the 0-simplex obtained by restriction of Σ^4 to the i'th vertex. Note that each $\overline{m}_i$ is degenerate by construction, so we have an induced morphism

$$F^{raw}(pr) : F^{raw}(\overline{m}_i) \to F^{raw}(x_0),$$

for pr the degeneracy morphism in $\Delta(S^4)$:

$$pr : \overline{m}_i \to \Sigma_i^0.$$

Finally, for each L_0^{i-1}, L_0^i we have a c-isomorphism

$$\gamma_i : L_0^{i-1} \to L_0^i$$

in $F^{raw}(\overline{m}_i) \subset F^{raw}(\Sigma_+)$, which corresponds to γ, meaning that the fully-faithful projection $F^{raw}(pr)$ takes γ_i to γ. We will denote by $\gamma_{i,j}$ the analogous c-isomorphisms $L_0^i \to L_0^j$.

Notation 5.2. *Let us denote from now on, the morphism spaces $hom_{F^{raw}(\Sigma_\pm)}(L_0, L_1)$ by $hom_{\Sigma_\pm}(L_0, L_1)$. And denote the A_∞ composition maps $\mu^d_{F^{raw}(\Sigma_\pm)}$, in the A_∞ category $F^{raw}(\Sigma_\pm)$, by $\mu^d_{\Sigma_\pm}$.*

Definition 5.3. *We call perturbation data $\mathcal{D}$ for P* **unexcited** *if it is extends the data $\mathcal{D}_{pt}$ as above, and if with respect to $\mathcal{D}$*

$$\mu^d_{\Sigma_+}(\gamma^1, \ldots, \gamma^d) = 0, \text{ for } 2 < d < 4, \tag{5.4}$$

where $(\gamma^1, \ldots, \gamma^d)$ is a composable chain, and each γ^k is of the form $\gamma_{i,j}$ as above.

We will see further on how to construct such unexcited data, assume for now that it is constructed.

Let $\{f_J\}$, corresponding to an n-simplex, be as in the definition of the A_∞ nerve in Appendix A.4 Part I, where J is a subset of $[n] = \{0, \ldots, n\}$.

Lemma 5.4. *Let $\mathcal{D}$ be unexcited as above, then there is a 4-simplex $\sigma \in NF^{raw}(\Sigma_+)$ with faces determined by the conditions:*

- *$f_J = 0$, for J any subset of $[4]$ with at least 3 elements.*

- $f_{\{i-1,i\}} = \gamma_i$ *for* γ_i *as before.*

Proof. This follows by (5.4) and by the identity $\mu^2_{\Sigma_+}(\gamma, \gamma) = \gamma$. □

If we take our unital replacements so that γ corresponds to the unit, then σ induces (by the construction) a section of $K(P_+) \to D_+$, where $K(P_\pm)$ will be shorthand for $K(P)$ restricted over $D_\pm$.

Let

$$i : \left(K(P_+)|_{\partial D_+} := p_\bullet^{-1}(\partial D_+)\right) \to K(P_-),$$

be the natural inclusion map. Set

$$sec = i \circ \sigma \circ h_+|_{\partial D^{4,mod}_\bullet}.$$

5.2. **The main lemma and immediate consequences.**

Lemma 5.5. *Suppose that P is a non-trivial Hamiltonian fibration and $\mathcal{D}$ is unexcited data for P as above, then sec as above does not extend to a section of $K(P_-)$. Moreover, unexcited data $\mathcal{D}$ exists.*

This lemma involves all the ingredients of our theory, its proof that will be broken up in parts, and will follow shortly.

Proof of Theorem 1.10. Clearly q-obs$(P) \geq 4$, since the 3-skeleton of S^4 is trivial. By Lemma 5.5 above, $K(P)$ does not have a section over the 4-skeleton. □

Remark 5.6. *When P is obtained by clutching with a generator of $\pi_3(PU(2))$, and when $h_\pm$ are embeddings, the class $[sec]$ in $\pi_3(K(P_-)) \simeq \pi_3(K(S^2))$ can be thought of as "quantum" analogue of the class of the classical Hopf map.*

Proof of Theorem 4.5. It might be helpful to first review Appendix A before reading the following. If we take any unexcited perturbation data $\mathcal{D}$ for P, then the first part follows immediately by Lemma 5.5. So $K(P)$ is non-trivial as a Kan fibration.

This then implies that $\mathrm{Fuk}_\infty(P)$ is non-trivial as a Cartesian fibration. To see this, suppose otherwise that we have a Cartesian fibration

$$\widetilde{\mathcal{P}} \to S^4_\bullet \times I_\bullet,$$

restricting to $\mathrm{Fuk}_\infty(P)$ over $S^4_\bullet \times 0_\bullet$ and to $\mathrm{NFuk}(S^2) \times S^4_\bullet$ over the other end $S^4_\bullet \times 1_\bullet$. Here $0_\bullet$, respectively $1_\bullet$ are notation for the images of $i_{j,\bullet} : \Delta^0_\bullet \to \Delta^1_\bullet$, $j = 0, 1$, where $i_{j,\bullet}$ are induced by the pair of boundary point inclusions.

Now take the maximal Kan sub-fibration of $\widetilde{\mathcal{P}}$, then by Lemma 4.2 we obtain a trivialization of $K(P)$ which is a contradiction. □

6. Towards the proof of Lemma 5.5

We will denote by $L_{0,\bullet}$ the image of the map $\Delta^0_\bullet \to K(P_-)$, induced by the inclusion of L_0 into $K(S^2)$ as a 0-simplex. Suppose that *sec* extends to a section of $K(P_-)$, so we have map

$$e : D^{4,mod}_\bullet \to K(P_-)$$

extending *sec* over $\partial D^{4,mod}_\bullet$. We may assume WLOG that e lies over h_-, meaning

$$p_\bullet \circ e = h_-,$$

since it can be homotoped to have this property. To see the latter, first take a relative homotopy of

$$p_\bullet \circ e : (D^{4,mod}_\bullet, \partial D^{4,mod}_\bullet) \to (D_-, \partial D_-)$$

to h_-, using that we have a homotopy equivalence of pairs (5.2), and then lift the homotopy to a relative homotopy upstairs using the defining lifting property of Kan fibrations.

And so we have a 4-simplex

$$T = e(\Sigma^4) \in K(P_-)$$

projecting to $\Sigma_- \in D_-$ by $p_\bullet$. Since T is in the image of e, all but one 3-faces of T are totally degenerate with image in $L_{0,\bullet}$. The exceptional 3-face is the sole non-degenerate 3-face of *sec*, (that is of $sec(\partial D^{4,mod}_\bullet)$).

Let $m_{i,j}, \gamma_{i,j}$ be as in the previous section, but corresponding now to Σ_- rather than Σ_+. Then by the boundary condition on e, the edges of T (which are all edges of *sec*) correspond, under the nerve construction, to the generators $\gamma_{i,j}$. As this is the condition for the edges of *sec*.

Lemma 6.1. *For $\mathcal{D}$ unexcited, and for the unital replacement F of F^{raw} as above, the simplex T exists if and only if $\mu^4_{\Sigma_-}(\gamma_1, \ldots, \gamma_4)$ is exact.*

Proof. The following argument will be over $\mathbb{F}_2$ as opposed to $\mathbb{Q}$ as the signs will not matter. Recall that we take the unital replacement so that $\gamma \in hom_{F^{raw}(P_{x_0})}(L_0, L_0)$ corresponds to the unit in the unital replacement.

Now if $T \in K(P_-)$ as above exists, then it corresponds under unital replacement (see Remark 7.5 in Part I) to a 4-simplex $T' \in NF^{raw}(\Sigma_-)$ satisfying the following condition on its 4-face. Recalling the nerve construction, the morphism $f_{[4]} \in hom_{\Sigma_-}(L^0_0, L^4_0)$, figuring in the definition of the 4-face, satisfies:

$$\mu^1_{\Sigma_-} f_{[4]} = \sum_{1<i<4} f_{[4]-i} + \sum_s \sum_{(J_1,\ldots J_s)\in decomp_s} \mu^s_{\Sigma_-}(f_{J_1}, \ldots, f_{J_s}). \tag{6.1}$$

By our conditions on the boundary of T, by the condition on the unital replacement, and by the conditions in Lemma 5.4, we must have $f_J = 0$, for every proper subset $J \subset [4]$, in some length s decomposition of [4], unless $J = \{i, j\}$ in which case $f_{i,j} = \gamma_{i,j}$. Given this (6.1) holds if and only if $\mu^4_{\Sigma_-}(\gamma_1, \ldots, \gamma_4)$ is exact.

∎

We are going to show that for some unexcited $\mathcal{D}$, $\mu^4_{\Sigma_-}(\gamma_1, \ldots, \gamma_4)$ does not vanish in homology, which will finish the proof of the Lemma 5.5.

7. Review of Hamiltonian structures

We review here the notion of a Hamiltonian structure. A complete formal definition of this is contained in [9, Definition 2.7]. A *Hamiltonian structure*

$$\{\widetilde{S}_r, S_r, \mathcal{L}_r, \mathcal{A}_r\}_{\mathcal{K}},$$

consists of the following data.

- A compact smooth manifold $\mathcal{K}$ possibly with boundary and corners (in the latter case $\mathcal{K}$ is meant to be a polyhedron so that there is no need for any deep theory of manifolds with corners).
- A suitably smooth $r \in \mathcal{K}$ family of Hamiltonian fibrations $M \hookrightarrow \widetilde{S}_r \to S_r$, where S_r are Riemann surfaces with boundary and with ends, s.t. each S_r is diffeomorphic to a closed disk with some number of punctures on the boundary. The data of a (r-dependent) holomorphic diffeomorphism at each i'th end,
$$e_i : [0,1] \times (0, \infty) \to S_r,$$
$i \neq 0$, (called positive ends). At the 0'th (called negative) end we ask for a holomorphic diffeomorphism
$$e_0 : [0,1] \times (-\infty, 0) \to S_r.$$
These charts are called *strip end charts*.
- The Hamiltonian fibrations $\widetilde{S}_r$ are endowed with Hamiltonian connections $\mathcal{A}_r$, which preserve a fiberwise Lagrangian subfibration $\mathcal{L}_r$ of $\widetilde{S}_r$ over ∂S_r. That is we have a fibration $\mathcal{L}_r \to \partial S_r$, which is embedded in $\widetilde{S}_r$, with fibers embedded as Lagrangian submanifolds of the fibers of $\widetilde{S}_r$.
- Trivializations
$$\widetilde{e}_i : [0,1] \times (0, \infty) \times M \to \widetilde{S}_r,$$
over e_i, called *end bundle charts* for $\widetilde{S}_r$.
- In the end bundle charts, the Hamiltonian connections $\mathcal{A}_r$ are flat and translation invariant. Specifically, they have the form $\overline{\mathcal{A}}_i$, where $\overline{\mathcal{A}}_i$ denotes the $\mathbb{R}$-translation invariant extension of some Hamiltonian connection $\mathcal{A}_i$ on $[0,1] \times M$ to $(0, \infty) \times [0,1] \times M$, (for the case of a positive end).
- A certain choice of a smooth family almost complex structures j_z on the fibers M_z of $\widetilde{S}_r$ (also r dependent). However, the latter will be implicit in the sense that we do not need to manipulate them, so we do not elaborate here, instead we refer the reader to [9, Definition 2.7].

For $\mathcal{A}_i$ as above, we say that $\mathcal{A}_r$ are *compatible* with $\{\mathcal{A}_i\}$. Let us write $(\widetilde{S}, \mathcal{L})$ for a pair as above.

7.1. **The Hamiltonian structures associated to the data $\mathcal{D}$.** As in Section 4 of Part I, $\Pi(\Delta^n)$ will denote a certain fundamental groupoid of Δ^n. More specifically, $\Pi(\Delta^n)$ is the small groupoid, whose objects set *obj* is the set of vertices of Δ^n. The morphisms set *hom* is the set of affine maps $m : [0,1] \to \Delta^n$, (possibly constant) sending end points to the vertices. The source map

$$s : hom \to obj$$

is defined by $s(m) = m(0)$ and the target map

$$t : hom \to obj$$

is defined by $t(m) = m(1)$.

Let $(m^1, \ldots, m^d)$ be a composable chain of morphisms in $\Pi(\Delta^n)$, which we recall means that the target of m^{i-1} is the source of m^i for each i. The perturbation data $\mathcal{D}$, in particular specifies for each n and for each such composable chain, certain maps

$$u(m^1, \ldots, m^d, n) : \mathcal{E}_d^\circ \to \Delta^n.$$

Here $\mathcal{E}_d$ is the universal curve over $\overline{\mathcal{R}}_d$, and $\mathcal{E}_d^\circ$ denotes $\mathcal{E}_d$ with nodal points of the fibers removed. The collection of these maps, satisfying certain axioms, is denoted by $\mathcal{U}$. We have already mentioned this in the introduction.

The restriction of $u(m^1, \ldots, m^d, n)$ to the fiber $\mathcal{S}_r$ of $\mathcal{E}_r^\circ$ over $r \in \overline{\mathcal{R}}_d$, is denoted by $u(m^1, \ldots, m^d, n, r)$ which may also be abbreviated by u_r.

Let $M \hookrightarrow P \to X$ be a Hamiltonian fibration with compact monotone fibers. Let $\Sigma : \Delta^n \to X$ be smooth, and denote $\overline{m}^i := \Sigma \circ m^i$. Suppose further we are given $u(m^1, \ldots, m^d, n)$ and a chain of Lagrangian branes (objects of the monotone Fukaya category $Fuk(M, \omega)$) $L'_0, \ldots, L'_d$ with $L'_i \subset P|_{\overline{m}_i(1)}$, $i \geq 1$, $L'_0 \subset P|_{\overline{m}_1(0)}$. Then the data $\mathcal{D}$ associates to this a Hamiltonian structure:

$$\{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r\}_{\overline{\mathcal{R}}_d}, \tag{7.1}$$

where:

- $\widetilde{\mathcal{S}}_r := (\Sigma \circ u_r)^* P$.
- $\widetilde{\mathcal{S}}_r$ is naturally trivial over the boundary components of $\mathcal{S}_r$. Likewise $\mathcal{L}_r$ is trivial over each i'th component of $\partial \mathcal{S}_r$, with fiber L'_i. Here the subscripts i correspond to the labels in the Figure 1.
- $\mathcal{A}_r$ are compatible with $\{\mathcal{A}_i = \mathcal{A}(L'_{i-1}, L'_i)\}$, with the later part of the data of the construction of the Fukaya category as in the preamble of Section 5.

8. Construction of unexcited data

Now, specialize the discussion of the section above to the case $n = 4$, $\Sigma = \Sigma_+$, $\forall i : L'_i = L_0$, with L_0 the distinguished equator as in (5.3). In this case, we write $\mathcal{A}_r^+$ for the corresponding connections, $(m_1, \ldots, m_d$ will be implicit).

Suppose that $\mathcal{D}$ extends $\mathcal{D}_{pt}$ from before. If $\mathcal{L}_r \subset \widetilde{\mathcal{S}}_r|_{\partial \mathcal{S}_r}$ denotes the trivial Lagrangian sub-bundle with fiber L_0, then we obtain a Hamiltonian structure

$$\Theta^+ = \{\Theta_r^+\} = \{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r^+\}_{\overline{\mathcal{R}}_d}.$$

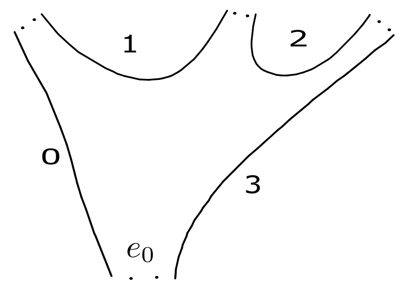


FIGURE 1.

To recall, as part of the requirements, at each end e_i of $\mathcal{S}_r$, $\mathcal{A}_r^+$ is compatible with the connection $\mathcal{A}_i = \mathcal{A}(L_0, L_0)$.

Set

$$\hbar := \frac{1}{2}\operatorname{Vol}(S^2, \omega).$$

Let κ denote the $L^{\pm}$-length of the holonomy path in $\operatorname{Ham}(S^2, \omega)$ of $\mathcal{A}_0 = \mathcal{A}(L_0, L_0)$. We may suppose that

$$\forall r : \text{energy}(\mathcal{A}_r^+) < \hbar - 5\kappa, \tag{8.1}$$

is satisfied after taking κ to be sufficiently small. (There is no obstruction since the corresponding bundles are naturally trivializeable, continuously in r.)

Fix a complex structure j_0 on M, and let $\{J_r = J(\mathcal{A}_r^+)\}$ be the corresponding, induced family of almost complex structures on $\{\widetilde{\mathcal{S}}_r\}$ as defined in [9, Section 2.5].

Let $\overline{\mathcal{M}}(\Theta^+, A)$ be as in [9, Section 2.5.1]. To paraphrase, this is the set of pairs (u, r) for u a relative homology class A, J_r-holomorphic, finite Floer energy section of $\widetilde{\mathcal{S}}_r$, with boundary on $\mathcal{L}_r$. The class A, again to paraphrase, is a relative class represented by an asymptotically $\mathcal{A}_r$-flat section of $\widetilde{S}_r$, with boundary on $\mathcal{L}_r$.

As in Part I, let

$$\overline{\mathcal{M}} = \overline{\mathcal{M}}(\gamma^1, \ldots, \gamma^d; \gamma^0, \Sigma_+, \{J_r\}, A),$$

denote the set of elements of $\overline{\mathcal{M}}(\Theta^+, A)$ with asymptotic constraints γ^i at each e_i end. Here each γ^k, $k \neq 0$, is of the form $\gamma_{i,j}$ where this is as in Section 5.1.

Lemma 8.1. *Whenever the class A is such that $\mathcal{M}$ has virtual dimension 0, and d satisfies $2 < d \leq 4$, $\overline{\mathcal{M}}$ is empty.*

Proof. Let

$$\Theta^{/} := (\Theta^+)^{/},$$

and $A^{/}$ be the capping off construction as in [9, Section 2.7]. For a fixed r, by the Riemann-Roch, [Appendix A][9], we get that the expected dimension of $\mathcal{M}(\Theta^{/}, A^{/})$ is

$$1 + Maslov^{vert}(A^{/}).$$

Consequently, when $\gamma^0 = \gamma$, the expected dimension of $\mathcal{M}$ is:

$$1 + Maslov^{vert}(A^{/}) - 1 + (\dim \mathcal{R}_d = d - 2). \tag{8.2}$$

We need the expected dimension of $\mathcal{M}$ to be 0, and $d \geq 3$, so $Maslov^{vert}(A^/) \leq -1$. But $Maslov^{vert}(A^/) = -1$ is impossible as the minimal positive Maslov number is 2.

Now, note that if $Maslov^{vert}(A^/) = -2$ then

$$-C \cdot Maslov^{vert}(A^/) = \hbar,$$

for C the monotonicity constant of (S^2, ω) and the equator L_0. Consequently, the result follows by [9, Lemma 2.34] and by the property (8.1).

When γ^0 is the Poincare dual to γ, we would get $Maslov^{vert}(A^/) \leq -2$ so for the same reason the conclusion follows. □

So if we choose our data $\mathcal{D}$ so that the hypothesis of the lemma above are satisfied, then with respect to this $\mathcal{D}$:

$$\mu^2_{\Sigma^4_\pm}(\gamma_{i,j}, \gamma_{j,k}) = \gamma_{i,k} \tag{8.3}$$

$$\mu^3_{\Sigma^4_\pm}(\gamma^1, \ldots, \gamma^3) = 0, \text{ for } \gamma^i \text{ as above} \tag{8.4}$$

$$\mu^4_{\Sigma^4_+}(\gamma_1, \ldots, \gamma_4) = 0. \tag{8.5}$$

In particular such $\mathcal{D}$ is unexcited, and $\mathcal{D}$ from now on will denote such a choice.

9. The product $\mu^4_{\Sigma^4_-}(\gamma_1, \ldots, \gamma_4)$ and the quantum Maslov classes

The product

$$\mu^4_{\Sigma^4_-}(\gamma_1, \ldots, \gamma_4), \tag{9.1}$$

a priori depends on various choices, like the choices of $h_\pm$, and then choice of data $\mathcal{D}$. For the purpose of computation we will take h_+ to be the constant map to x_0 and

$$h_- : (D^4, \partial D^4) \to (S^4, x_0)$$

to be the complementary map, that is representing the generator of $\pi_4(S^4, x_0) \simeq \mathbb{Z}$. We further suppose that h_- is an embedding in the interior of D^4.

We are then going to reduce the computation of the product (9.1) to the computation of a certain quantum Maslov class, already done in [9]. This will require a number of geometric steps.

Let Σ_- be the 4-simplex of $S^4_\bullet$ corresponding to h_- as before in Section 5.1. We need to study the moduli spaces

$$\overline{\mathcal{M}}(\gamma_1, \ldots, \gamma_4; \gamma^0, \Sigma_-, \{\mathcal{A}_r\}, A), \tag{9.2}$$

where $\mathcal{A}_r$ now denotes the connections on

$$\widetilde{\mathcal{S}}_r := (\Sigma_- \circ u(m_1, \ldots, m_4, 4, r))^* P \to \mathcal{S}_r, \tag{9.3}$$

part of the data $\mathcal{D}$. We abbreviate $u(m_1, \ldots, m_4, 4, r)$ by u_r in what follows.

By the dimension formula (8.2), since we need the expected dimension of (9.2) to be zero, the class $A^/$ satisfies:

$$Maslov^{vert}(A^/) = -2,$$

and we must have $\gamma^0 = \gamma_{0,4}$.

Notation 9.1. *From now on, by slight abuse, A_0 refers to various section classes of various Hamiltonian structures such that the associated class $A_0^{/}$ satisfies:*

$$Maslov^{vert}(A_0^{/}) = -2.$$

9.1. **Constructing suitable $\{\mathcal{A}_r\}$.** To get a handle on (9.2) we will construct unexcited data $\mathcal{D}_0$ with additional geometric properties.

A Hamiltonian S^2 fibration over S^4 is classified by an element

$$[g] \in \pi_3(\mathrm{Ham}(S^2, \omega), id) \simeq \pi_3(PU(2), id) \simeq \mathbb{Z}.$$

Such an element determines a fibration P_g over S^4 via the clutching construction:

$$P_g = D^4_- \times S^2 \sqcup D^4_+ \times S^2 \sim,$$

with D^4_-, D^4_+ being 2 different names for the standard closed 4-ball D^4, and where the equivalence relation $\sim$ is $(d, x) \sim \widetilde{g}(d, x)$,

$$\widetilde{g} : \partial D^4_- \times S^2 \to \partial D^4_+ \times S^2, \quad \widetilde{g}(d, x) = (d, g(d)^{-1}(x)).$$

We suppose that the that previously appeared point $x_0 = h_{\pm}(b_0)$, is in $D^4_+ \cap D^4_- \subset S^4$.

From now on P_g will denote such a fibration for a non-trivial class $[g]$. Note that the fiber of P_g over the base point $x_0 \in S^3 \subset D^4_{\pm}$ (chosen for definition of the homotopy group $\pi_3(\mathrm{Ham}(S^2, \omega), id)$) has a distinguished, by the construction, identification with S^2. Take $\mathcal{A}$ to be a connection on $P \simeq P_g$ which is trivial in the distinguished trivialization over D^4_+. This gives connections

$$\mathcal{A}'_r := (\widetilde{u}_r)^* \mathcal{A}$$

on $\widetilde{\mathcal{S}}_r$, where

$$\widetilde{u}_r = \Sigma_- \circ u_r.$$

By the last axiom for the system $\mathcal{U}$ introduced in Part I, we may choose $\{u_r\}$ so that the family $\{\widetilde{u}_r(\mathcal{S}_r)\}$ induces a singular foliation of S^4 with the properties:

- The foliation is smooth outside x_0. Note that x_0 is the image by $\widetilde{u}_r$ of the ends (images of e_i), and the image of the boundary of each $\mathcal{S}_r$.
- Each $\widetilde{u}_r$ is an embedding on the complement of $\widetilde{u}_r^{-1}(x_0)$.

Denote by E the subset $S^3 \subset S^4$ bounding $D^4_{\pm}$. We may in addition suppose that each $\widetilde{u}_r$ intersects E transversally, again on the complement of $\widetilde{u}_r^{-1}(x_0)$.

By the above, the preimage by $\widetilde{u}_r$ of E contains a smoothly embedded curve c_r as in Figure 2, and $\widetilde{u}_r$ takes c_r into E. This c_r not uniquely determined, but we may fix a family $r \mapsto c_r$, with parametrizations

$$c_r : \mathbb{R} \to \mathcal{S}_r,$$

with the properties:

- c_r maps $(-\infty, 0)$ diffeomorphically onto $e_0(\{0\} \times (-\infty, 0))$.
- c_r maps $(1, \infty)$ diffeomorphically onto $e_0(\{1\} \times (-\infty, 0))$.
- $\{c_r\}$ is a C^0 continuous family in r.

We set:

$$\widetilde{c}_r := \widetilde{u}_r \circ c_r.$$

In Figure 2, the regions $R_\pm$ are the preimages by $\widetilde{u}_r$ of $D^4_\pm \subset S^4$, and c_r bounds R_-. It follows that $\{\widetilde{c}_r\}$ likewise induces a singular foliation of the equator $E \simeq S^3$

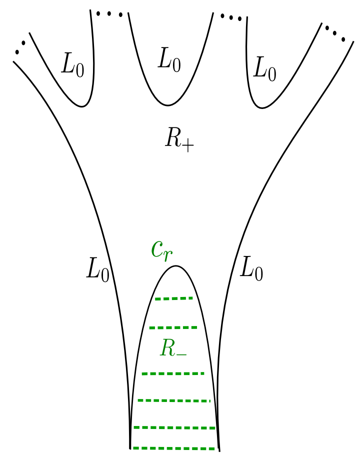


FIGURE 2. The labels L_0 indicate that the Lagrangian subbundle is constant with corresponding fiber L_0. The curve c_r bounds R_-.

that is smooth outside x_0.

So each $\mathcal{A}'_r$ is flat in the region R_+, in fact is trivial in the distinguished trivialization of $\widetilde{\mathcal{S}}_r$ over R_+, corresponding to the distinguished trivialization of P over D^4_+. Likewise, we have a distinguished trivialization of $\widetilde{\mathcal{S}}_r$ over R_-, corresponding to the distinguished trivialization of P over D^4_-. In this latter trivialization let

$$\phi_r : \mathbb{R} \to \mathrm{Ham}(S^2, \omega)$$

be the holonomy path of $\mathcal{A}'_r$ over c_r.

Let $Lag(S^2)$ denote the space of Lagrangian equators, that is oriented Lagrangian submanifolds Hamiltonian isotopic to the standard oriented equator. Then by construction,

$$\phi_r|_{(-\infty,0] \sqcup [1,\infty)} = id,$$

so that we may define

$$f(r) \in \Omega_{L_0} Lag(S^2))$$

by

$$f(r)(t) = \phi_r(t) \cdot L_0, \quad t \in [0,1] \tag{9.4}$$

where the right hand side means apply an element of $\mathrm{Ham}(S^2, \omega)$ to L_0 to get a new Lagrangian.

Terminology 9.2. *We will say that $f(r)$ is* **generated by** $\mathcal{A}'_r$.

Note that each $f(r)$ is an exact loop by construction, where exact is the standard notion, as for example in [9, Section 4.1.1].

Also by construction

$$\phi_r(t) = g(\widetilde{c}_r(t)), \quad t \in [0,1], \tag{9.5}$$

if we identify $\widetilde{c}_r(t)$ with an element of S^3.

Let $D_0^2 \subset \overline{\mathcal{R}}_4$ be an embedded closed disk, not intersecting the boundary $\partial\overline{\mathcal{R}}_4$, so that ∂D_0^2 is in the gluing normal neighborhood N of $\partial\overline{\mathcal{R}}_4$, as defined in Part I.

So we get a continuous map

$$f : D_0^2 \to \Omega_{L_0} Eq(S^2),$$

with $Eq(S^2) \simeq S^2$ denoting the space of standard oriented equators in S^2. And $f(\partial D_0^2) = p_{L_0}$, with the right-hand side denoting the constant loop at L_0. Then by construction, and (9.5) in particular, $f \simeq lag$, where $\simeq$ is a homotopy equivalence, and where

$$lag : S^2 \to \Omega_{L_0} Lag(S^2) \tag{9.6}$$

is the composition

$$S^2 \xrightarrow{g'} \Omega_{id} PU(2) \to \Omega_{L_0} Eq(S^2),$$

for g' naturally induced by g, and for the second map naturally induced by the map

$$PU(2) \to Eq(S^2), \quad \phi \mapsto \phi(L_0).$$

We then deform each $\mathcal{A}'_r$ to a connection $\mathcal{A}_r$, which is as follows. In the region R_+ $\mathcal{A}_r$ is still flat, but at each end e_i, $\mathcal{A}_r$ is compatible with $\mathcal{A}(L_0, L_0)$, and so that $\mathcal{A}_r$ is still trivial over the boundary of $\mathcal{S}_r$.

Since $\widetilde{\mathcal{S}}_r$ and $\mathcal{A}'_r$ are trivial for $r \in \overline{\mathcal{R}}_d - D_0^2$, with trivialization induced by the trivialization of P_+, and since the condition (8.1) holds, we may ensure that

$$\text{energy}(\mathcal{A}_r) < \hbar - 5\kappa, \tag{9.7}$$

for r in the complement of D_0^2. This give a Hamiltonian structure we denote as

$$\mathcal{H} := \{\mathcal{H}_r := (\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}_r)\}.$$

By the paragraph after Lemma 8.1, this extends to some unexcited data $\mathcal{D}_0$.

9.2. **Restructuring $\mathcal{H}$.** The data $\mathcal{D}_0$ is now fixed, but it remains compute

$$\mu^4_{\Sigma^4_-}(\gamma_1, \ldots, \gamma_4),$$

with respect to this data. To this end we are going to restructure $\mathcal{H}$.

For convenience we recall here some notions from [9].

Definition 9.3. *Given a pair* $\{\Theta^i_r\} = \{\widetilde{S}^0_r, S^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}_{\mathcal{K}}$, *of Hamiltonian structures we say that they are* **concordant** *if the following holds. There is a Hamiltonian structure*

$$\mathcal{T} = \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K}\times[0,1]},$$

with an oriented diffeomorphism (in the natural sense, preserving all structure)

$$\{\widetilde{S}^0_r, S^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}_{\mathcal{K}_0^{op}} \sqcup \{\widetilde{S}^1_r, S^1_r, \mathcal{L}^1_r, \mathcal{A}^1_r\}_{\mathcal{K}_1} \to \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K}\times\partial[0,1]}.,$$

where op denotes the opposite orientation.

Definition 9.4. *We say that a Hamiltonian structure $\{\Theta_r\}$ is A-**admissible** if there are no elements*

$$(\sigma, r) \in \overline{\mathcal{M}}(\{\Theta_r\}, A),$$

for r in a neighborhood of the boundary of $\mathcal{K}$.

Definition 9.5. *Given an A-admissible pair $\{\Theta^i_r\}$, $i = 1, 2$, of Hamiltonian structures, we say that they are A-**admissibly concordant** if there is a Hamiltonian structure*

$$\{\mathcal{T}_r\} = \{\widetilde{T}_r, T_r, \mathcal{L}'_r, \mathcal{A}'_r\}_{\mathcal{K}\times[0,1]},$$

which furnishes a concordance, and s.t. there are no elements $(\sigma, r) \in \overline{\mathcal{M}}(\{\Theta_r\}, A)$, for $r \in \partial\mathcal{K} \times [0, 1]$.

Applying [9, Lemma 2.34], and using (9.7) we get that $\mathcal{H}$ is A_0-admissible. We now further mold this data for the purposes of computation.

First cap off the ends e_i, $i \neq 0$, of each $\mathcal{H}_r$ as in the paragraph preceding [9, Lemma 2.34]. This gives a Hamiltonian structure

$$\mathcal{H}^\wedge := \{\widetilde{S}^\wedge_r, S^\wedge_r, \mathcal{L}^\wedge_r, \mathcal{A}^\wedge_r\}_{\mathcal{K}=D^2_0},$$

satisfying

$$\text{energy}(\mathcal{A}^\wedge_r) + \kappa < \hbar, \tag{9.8}$$

for each r. Again by [9, Lemma 2.34] $\mathcal{H}^\wedge$ is A_0-admissible.

Let

$$ev(\mathcal{H}^\wedge, A_0) \in CF(L_0, L_0, \mathcal{A}(L_0, L_0))$$

be as in [9, Lemma 2.14].

Using the PSS maps as described in [9, Section 2.7.1], corresponding to the caps at the ends (needed for the construction $\mathcal{H}^\wedge$), and using standard gluing, it readily follows that

$$[\mu^4_{\Sigma^4_-}(\gamma_1, \ldots, \gamma_4)] = [ev(\mathcal{H}^\wedge, A_0)] \in FH(L_0, L_0). \tag{9.9}$$

It remains to compute the right-hand side, to this end we further restructure the data.

Let $p_1 : [0, 1] \to Lag(S^2)$ be the path generated by $\mathcal{A}(L_0, L_0)$, with p_1 starting at L_0, and where generated is as in the paragraph following (9.4). Suppose we have defined p_{i-1}. Set $L_{i-1} := p_{i-1}(1)$ and define p_i to be the path in $Lag(S^2)$ starting at L_{i-1}, generated by $\mathcal{A}(L_0, L_0)$. Now set $p_0 := p_1 \cdot \ldots \cdot p_d$, where $\cdot$ is path concatenation in diagrammatic order. We may assume that L_0 is transverse to $L_4 = p_0(1)$ by adjusting the connection $\mathcal{A}(L_0, L_0)$ if necessary.

We construct a concordance of $\mathcal{H}^\wedge$ to another Hamiltonian structure:

$$\mathcal{H}^\mathfrak{n} := \{\widetilde{S}^\wedge_r, S^\wedge_r, \mathcal{L}^\mathfrak{n}_r, \mathcal{A}^\mathfrak{n}_r\},$$

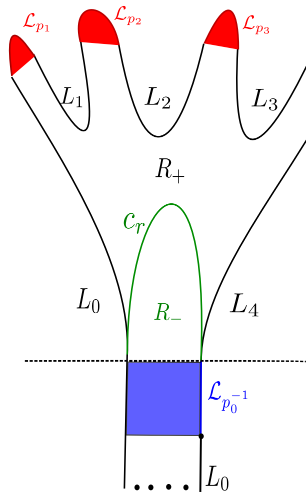


FIGURE 3. Over the boundary components with black labels L_i the Lagrangian subbundle $\mathcal{L}_r^{\mathfrak{n}}$ is constant with corresponding fiber L_i. Over the i'th red boundary component the Lagrangian subbundle corresponds to the path of Lagrangians p_i. Likewise over the right boundary component of the blue region, the Lagrangian subbundle corresponds to the path of Lagrangians p_0^{-1}. In the red striped regions we have removed the curvature of the connection, in the blue striped region we have added it.

whose properties are illustrated in Figure 3.

The Hamiltonian connection $\mathcal{A}_r^{\mathfrak{n}}$ satisfies the following conditions, (referring to the Figure 3):

- $\mathcal{A}_r^{\mathfrak{n}}$ is flat in the entire region R_+ (which includes the red shaded finger regions).
- The blue region is contained in the strip end chart at the e_0 end, which is down in the figure.
- Along top boundary segment of the blue region, contained in the dashed line, $\mathcal{A}_r^{\mathfrak{n}}$ is the trivial connection in the coordinates of corresponding end bundle chart.
- At the e_0 end, in the corresponding end bundle chart coordinates, the connection $\mathcal{A}_r^{\mathfrak{n}}$ coincides with $\mathcal{A}_r^{\wedge}$ over $[0,1] \times (-\infty, s)$, for some $s < 0$.

The connections $\mathcal{A}_r^{\mathfrak{n}}$ are forced by the above conditions to gain curvature in some region, and we limit this region to the blue region of Figure 3. This works more specifically as follows.

We choose a concordance $\mathcal{T}$ from $\mathcal{H}^{\wedge}$ to $\mathcal{H}^{\mathfrak{n}}$ such that for the associated family of connections $\{\mathcal{A}_{r,t}\}$, $t \in [0,1]$, the following is satisfied:

(1)
$$\mathcal{A}_{r,0} = \mathcal{A}_r^{\wedge}, \quad \mathcal{A}_{r,1} = \mathcal{A}_r^{\mathfrak{n}}.$$

(2)
$$\forall r \, \forall v \in T_z \mathcal{S}_r : \frac{d}{dt} |R_{\mathcal{A}_{r,t}}(v, jv)|_+ < 0,$$
for each $z \in \mathcal{S}_r$, except for z in the region which is blue shaded in Figure 3.

(3)
$$\forall t : |\operatorname{energy}(\mathcal{A}_{r,t}) - \operatorname{energy}(\mathcal{A}_r^{\wedge})| = 0.$$

The last condition can be satisfied, since the gain of energy in the blue region is exactly equal to the loss of energy in the red regions.

It follows by the conditions above, and the use of [9, Lemma 2.34] that $\mathcal{T}$ is an A_0-admissible concordance.

By [9, Lemma 2.13] we have:
$$[ev(\mathcal{H}^{\wedge}, A_0)] = [ev(\mathcal{H}^{\mathfrak{n}}, A_0)].$$
This finishes our restructuring.

9.3. **Computing** $[ev(\mathcal{H}^{\mathfrak{n}}, A_0)]$**.** If we stretch the neck along the dashed line in Figure 3, the upper half of the resulting building gives us a new Hamiltonian structure:
$$\mathcal{H}^0 = \{\widetilde{S}_r^0, S_r^0, \mathcal{L}_r^0, \mathcal{A}_r^0\}.$$

Let
$$[ev_{\mathfrak{n}}] := [ev(\mathcal{H}^{\mathfrak{n}}, A_0)] \in FH(L_0, L_0),$$
and let
$$[ev_0] := [ev(\mathcal{H}^0, A_0] \in FH(L_0, L_4).$$
By standard Floer theory, and specifically the theory of continuation maps in Floer homology we clearly have that:
$$0 \neq [ev_{\mathfrak{n}}] \iff 0 \neq [ev_0].$$

9.3.1. *Getting a cycle of Lagrangian paths.* We may suppose that the holonomy path of $\mathcal{A}_r^0$ over c_r, in the distinguished trivialization over R_+, generates p_0. Since $\mathcal{A}^0$ is flat over R_+, the latter can be insured simply by adjusting the parametrizations $\{c_r\}$.

Let $\mathcal{P}(L_0, L_4)$ denote the space of smooth exact paths in $Lag(S^2)$ from L_0 to L_4. Let
$$f' : D_0^2 \to \mathcal{P}(L_0, L_4),$$
be defined as:
$$f'(r)(t) = g(\widetilde{c}_r(t)) \cdot p_0(t), \tag{9.10}$$
Here the right-hand side of (9.10) means the action as in (9.4): apply an element of $\operatorname{Ham}(S^2, \omega)$ to a Lagrangian to get a new Lagrangian. In this case
$$f'(\partial D_0^2) = p_0 \in \mathcal{P}(L_0, L_4).$$

In particular, f' represents a class

$$a \in \pi_2(\mathcal{P}(L_0, L_4), p_0) \tag{9.11}$$

In what follows, we omit specifying the parameter space D^2_0 for r, since it will be the same everywhere.

Let $\mathcal{O}$ denote the Riemann surface with one end, diffeomorphic to the closed disk with a single puncture on the boundary. Let $\mathcal{L}_p$ be the Lagrangian sub-fibration of $\mathcal{O} \times S^2$, over the boundary induced by p, and defined as follows.

as formally defined in [9, Definition 2.27]. F

Lemma 9.6. *The A_0-admissible Hamiltonian structure $\mathcal{H}^0 = \{\widetilde{\mathcal{S}}^0_r, \mathcal{S}^0_r, \mathcal{L}^0_r, \mathcal{A}^0_r\}$ is A_0-admissibly concordant to*

$$\Theta' = \{\mathcal{O} \times S^2, \mathcal{O}, \mathcal{L}_{f'(r)}, \mathcal{B}_r\},$$

for certain Hamiltonian connections $\{\mathcal{B}_r\}$ made explicit in the proof.

Proof. Let $R_\pm \subset \mathcal{S}^\wedge_r$ be as before. Fix a family of smooth deformation retractions

$$ret_r : \mathcal{S}^0_r \times I \to \mathcal{S}^0_r,$$

of $\mathcal{S}^0_r$ onto R^-, smooth in r, with $ret_0 = id$. We may use the smooth Riemann mapping theorem to identify each $R^- \subset \mathcal{S}^0_r$ with its induced complex structure j_r with $(\mathcal{O}, j_{st})$, smoothly in r.

Set $ret_{r,t} = ret_r|_{\mathcal{S}^0_r \times \{t\}}$, set $\mathcal{S}^t_r = \text{image}\, ret_{r,t}$ so that $\mathcal{S}^1_r = R_-$, set $\widetilde{\mathcal{S}}^t_r = \widetilde{\mathcal{S}}^0_r|_{\mathcal{S}^t_r}$, i.e. the pull-back under inclusion of $\mathcal{S}^t_r$. Next set $\mathcal{A}^t_r = i^*_{r,t}\mathcal{A}^0_r$ where $i_{r,t} : \widetilde{\mathcal{S}}^t_r \to \widetilde{\mathcal{S}}^0_r$ is the inclusion.

Let $\mathcal{L}^t_r$ be the Lagrangian sub-bundle uniquely determined by the following conditions:

(1) In the end bundle chart of $\widetilde{\mathcal{S}}^t_r$, the fiber of $\mathcal{L}^t_r$ over $\{0\} \times \{t\} \subset [0,1] \times (-\infty, 0)$ is L_0. The fiber of $\mathcal{L}^t_r$ over $\{1\} \times \{t\} \subset [0,1] \times (-\infty, 0)$ is L_4.

(2) $\mathcal{L}^t_r$ is preserved by $\mathcal{A}^t_r$.

Existence of $\mathcal{L}^t_r$ as above, is guaranteed by the flatness of $\mathcal{A}^0_r$ over R^+.

We then get a Hamiltonian structure

$$\widetilde{\mathcal{H}} = \{\widetilde{\mathcal{S}}^t_r, \mathcal{S}^t_r, \mathcal{L}^t_r, \mathcal{A}^t_r\}_{r,t}.$$

Given the Riemann mapping argument above, $\{\mathcal{S}^1_r \simeq R_-\} \simeq \{\mathcal{O}\}$ smoothly in r. So $\widetilde{\mathcal{H}}$ is a concordance between $\mathcal{H}^0$ and

$$\{\mathcal{O} \times S^2, \mathcal{O}, \mathcal{L}_{f'(r)}, \mathcal{B}_r\},$$

where $\mathcal{B}_r$ is $\mathcal{A}^1_r$ under identifications. Finally, the usual application of [9, Lemma 2.34] gives that $\widetilde{\mathcal{H}}$ is an A_0-admissible concordance. ∎

10. Finishing up the proof of Lemma 5.5

The existence of unexcited data $\mathcal{D}$ is proved in Section 8. Given this existence, starting with (9.9) we showed that $[\mu^4_{\Sigma^4}(\gamma_1, \ldots, \gamma_4)]$ is non-vanishing in Floer homology iff

$$[ev(\mathcal{H}_0, A_0)] \in HF(L_0, L_4),$$

is non-vanishing. We then use Lemma 9.6 to identify $[ev(\mathcal{H}_0, A_0)]$ with $[ev(\Theta', A_0)]$.

Now by definitions

$$[ev(\Theta', A_0)] = \Psi(a),$$

where the right hand side is the quantum Maslov class as defined in [9], and where a is a spherical class as in (9.11). Given that g is non-trivial the corresponding class a is non-trivial and so $\Psi(a)$ is non-trivial by Theorem [9, Theorem 4.13]. This together with Lemma 6.1 imply Lemma 5.5. □

11. Singular and simplicial connections and curvature bounds

Let $\mathcal{A}$ be a G connection on a principal G bundle $P \to \Delta^n$, and the Finsler norm $\mathfrak{n}$ on $\operatorname{lie} G$ be as in Section 1.2.1 of the introduction. As previously discussed, a given system $\mathcal{U}$ in particular specifies maps:

$$u(m_1, \ldots, m_n, r, n) : \mathcal{S}_r \to \Delta^n,$$

where $r \in \overline{\mathcal{R}}_n$, $\mathcal{S}_r$ is the fiber of $\mathcal{S}^\circ_n$ over r, and where $(m_1, \ldots, m_n)$ is the composable chain of morphisms in $\Pi(\Delta^n)$, m_i being the edge morphism from the vertex $i-1$ to i. Then define

$$\operatorname{energy}_{\mathcal{U}}(\mathcal{A}) = \sup_r \operatorname{energy}_{\mathfrak{n}}(u(m_1, \ldots, m_n, r, n)^* \mathcal{A}), \tag{11.1}$$

where $\operatorname{energy}_{\mathfrak{n}}$ on the right hand side is as defined in equation (1.2). In the case $G = \operatorname{Ham}(M, \omega)$ we take

$$\mathfrak{n} : \operatorname{lie} \operatorname{Ham}(M, \omega) \to \mathbb{R}$$

to be

$$\mathfrak{n}(H) = |H|_+ = \max_M H.$$

Let ω be the area 1 Fubini-Study symplectic 2-form on $M = \mathbb{CP}^1$. Then the pull-back by the natural map

$$\operatorname{lie} h : \operatorname{lie} PU(2) \to \operatorname{lie} \operatorname{Ham}(\mathbb{CP}^1, \omega) \simeq C^\infty_0(\mathbb{CP}^1)$$

of the semi-norm: $|H|_+ = \max_M H$ is the operator norm on $PU(2)$, up to normalization. This will be used to get the specific form of Theorem 1.7, from the more general form here.

11.1. Simplicial connections.

We now introduce the notion of simplicial connections, which can partly be understood as simplicial resolutions of singular connections. Let $G \hookrightarrow P \to X$ be a principal G bundle, where G is a Frechet Lie group. Denote by $X_\bullet$ the smooth singular set of X, i.e. the simplicial set whose set of n-simplices, $X_\bullet(n)$ consists smooth maps $\Sigma : \Delta^n \to X$, with Δ^n standard topological

n-simplex with vertices ordered $0, \ldots, n$. And denote by $Simp(X_\bullet)$ the category with objects $\cup_n X_\bullet(n)$ and with $hom(\Sigma_0, \Sigma_1)$ commutative diagrams:

$$\begin{array}{ccc} \Delta^n & \xrightarrow{mor} & \Delta^m \\ & \searrow^{\Sigma_0} & \downarrow{\scriptstyle \Sigma_1} \\ & & X, \end{array}$$

for mor a simplicial face map, that is an injective affine map preserving order of the vertices.

Definition 11.1. *Define a* **simplicial G-connection** $\mathcal{A}$ *on* P *to be the following data:*

- *For each* $\Sigma : \Delta^n \to X$ *in* $X_\bullet(n)$ *a smooth* G*-connection* $\mathcal{A}_\Sigma$ *on* $\Sigma^* P \to \Delta^n$*, (a usual Ehresmann* G*-connection.)*
- *For a morphism* $mor : \Sigma_0 \to \Sigma_1$ *in* $Simp(X_\bullet)$*, we ask that* $mor^* \mathcal{A}_{\Sigma_1} = \mathcal{A}_{\Sigma_0}$.

Example 11.2. *If* $\mathcal{A}$ *is a smooth* G*-connection on* P*, define a simplicial connection by* $\mathcal{A}_\Sigma = \Sigma^* \mathcal{A}$ *for every simplex* $\Sigma \in X_\bullet$*. We call such a simplicial connection* **induced**.

If we try to "push forward" a simplicial connection to get a "classical" connection on P over X, then we get a kind of multi-valued singular connection. Multi-valued because each $x \in X$ may be in the image of a number of $\Sigma : \Delta^n \to X$ and Σ itself may not be injective, and singular because each Σ is in general singular so that the naive push-forward may have blow up singularities. We will call the above the naive pushforward of a simplicial connection.

Proof of Theorem 1.7 and Corollary 1.3. We will prove this by way of a stronger result. Let P be a Hamiltonian fibration $S^2 \hookrightarrow P \to S^4$, and $\mathcal{A}$ a simplicial $\mathcal{G} = \mathrm{Ham}(S^2, \omega)$ connection on P. Denote by $\sigma_0^1 \in S^4_\bullet$ the degenerate 1-simplex at x_0, in other words the constant map: $\sigma_0^1 : [0,1] \to x_0$. Let κ be the $L^\pm$-length of the holonomy path of $\mathcal{A}_{\sigma_0^1}$ over $[0,1]$.

Finally, let $\Sigma_\pm \in S^4_\bullet(4)$ be a complementary pair as in Section 5.1. The connection $\mathcal{A}$ gives us a simplicial connection as in Example 11.2. By an inductive procedure as in Part I, Lemma 5.6, we may find perturbation data $\mathcal{D}$ for P so that with respect to $\mathcal{D}$ the following is satisfied.

$$\forall r : pr_1 \mathcal{F}(L_0^0, \ldots, L_0^n, \Sigma_\pm, r) \simeq_\delta u(m_1, \ldots, m_s, r, n)^* \mathcal{A}_{\Sigma_\pm}, \tag{11.2}$$

$$\mathcal{A}(L_0, L_0) \simeq_\delta \mathcal{A}_{\sigma_0^1}, \tag{11.3}$$

where L_0^i are the objects as before, where $\simeq_\delta$ means δ-close in the metrized C^∞ topology, and δ is as small as we like. Here we are using notation of Part I as before. Set

$$\widetilde{u}_r := \Sigma_- \circ u(m_1, \ldots, m_4, r, 4),$$

so $\widetilde{u}_r : \mathcal{S}_r \to S^4$. Set $\widetilde{\mathcal{S}}_r := \widetilde{u}_r^* P$, set $\mathcal{A}'_r := pr_1 \mathcal{F}(L_0^0, \ldots, L_0^n, \Sigma_-, r)$ and set

$$\{\Theta_r\} := \{\widetilde{\mathcal{S}}_r, \mathcal{S}_r, \mathcal{L}_r, \mathcal{A}'_r\}.$$

Definition 11.3. *Let δ and $\mathcal{A}(L_0, L_0)$ be as above. We say that $\mathcal{A}$ is* **perfect** *if the following holds. For every arbitrarily small δ $\mathcal{A}(L_0, L_0)$ can be chosen so that the corresponding Floer chain complex $CF(L_0, L_1, \mathcal{A}(L_0, L_0))$ is perfect.*

Theorem 11.4. *Let $\mathcal{A}$ be a perfect simplicial Hamiltonian connections on P. If P is non-trivial as a Hamiltonian bundle then*

$$(\text{energy}_{\mathcal{U}}(\mathcal{A}_{\Sigma_+}) \geq \hbar - 5\kappa) \vee (\text{energy}_{\mathcal{U}}(\mathcal{A}_{\Sigma_-}) \geq \hbar - 5\kappa),$$

Proof. Suppose

$$\text{energy}_{\mathcal{U}}(\mathcal{A}_{\Sigma_+}) < \hbar - 5\kappa. \tag{11.4}$$

Then by (11.2), (11.2) and by [9, Lemma 2.34] $\mathcal{D}$, as defined above, can be assumed to be unexcited provided δ is chosen to be sufficiently small. Take the unital replacement as in Lemma 6.1. Since we know that $K(P)$ does not admit a section by Theorem 4.5, the simplex T of the Lemma 6.1 does not exist. Hence, again by this lemma

$$ev(\{\Theta_r\}, A_0) = [\mu^4_{\Sigma_-}(\gamma_1, \ldots, \gamma_4)] \neq 0.$$

In particular

$$\overline{\mathcal{M}}(\{\Theta_r\}, A_0) \neq \emptyset.$$

So by [9, Lemma 2.34] there exists an r_0 so that

$$\text{energy}(\mathcal{A}'_{r_0}) \geq \hbar - 5\kappa'. \tag{11.5}$$

where κ' denotes the $L^{\pm}$ length of the holonomy path in $\text{Ham}(S^2, \omega)$ of $\mathcal{A}(L_0, L_0)$. By (11.3) $\kappa' \to \kappa$ as $\delta \to 0$. By (11.2), (11.5) passing to the limit as $\delta \to 0$ we get:

$$\text{energy}_{\mathcal{U}}(\mathcal{A}_{\Sigma_-}) \geq \hbar - 5\kappa.$$

□

Corollary 11.5. *Let $\mathcal{A}$ be a $PU(2)$ connection on a non-trivial principal $PU(2)$ bundle $P \to S^4$. Then*

$$(\text{energy}_{\mathcal{U}}(\mathcal{A}_{\Sigma_+}) \geq \hbar - 5\kappa) \vee (\text{energy}_{\mathcal{U}}(\mathcal{A}_{\Sigma_-}) \geq \hbar - 5\kappa).$$

Proof. A simplicial $PU(2)$ connection $\mathcal{A}$ on a principal $PU(2)$ bundle $PU(2) \hookrightarrow P' \to S^4$ is automatically perfect, when understood as a Hamiltonian connection on the associated bundle $S^2 \hookrightarrow P \to S^4$. So that this is an immediate consequence of the theorem above. □

To prove Corollary 1.3, we just note that for an induced simplicial Hamiltonian connection $\mathcal{A}$, as defined in Example 11.2, $\mathcal{A}_{\sigma^1_0}$ is trivial. And hence $\mathcal{A}$ is automatically perfect. So that this corollary follows by Theorem 11.4.

□

Appendix A. Homotopy groups of Kan complexes

For convenience let us quickly review Kan complexes just to set notation. This notation is also used in Part I. Let

$$\Delta^n_\bullet(k) := hom_\Delta(k, n),$$

be the standard representable n-simplex, where Δ is as in Section 3.

Let $\Lambda^n_k \subset \Delta^n_\bullet$ denote the sub-simplicial set corresponding to the "boundary" of $\Delta^n_\bullet$ with the k'th face removed, $0 \leq k \leq n$. By $k'th$ face we mean the face opposite to the k'th vertex. Let $X_\bullet$ be an abstract simplicial set. A simplicial map

$$h : \Lambda^n_k \subset \Delta^n_\bullet \to X_\bullet$$

will be called a ***horn***. A simplicial set $S_\bullet$ is said to be a ***Kan complex*** if for all $n, k \in \mathbb{N}$ given a diagram with solid arrows

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{h} & S_\bullet \\ \downarrow{\scriptstyle i} & \overset{\widetilde{h}}{\nearrow} & \\ \Delta^n_\bullet & & \end{array},$$

there is a dotted arrow making the diagram commute. The map $\widetilde{h}$ will be called ***the Kan filling*** of the horn h. The k'th face of $\widetilde{h}$ will be called ***Kan filled face along*** h. As before we will denote Kan complexes and ∞-categories by calligraphic letters.

Given a pointed Kan complex $(\mathcal{X}, x)$ and $n \geq 1$ the *n'th simplicial homotopy group* of $(\mathcal{X}, x)$: $\pi_n(\mathcal{X}, x)$ is defined to be the set of equivalence classes of maps

$$\Sigma : \Delta^n_\bullet \to \mathcal{X},$$

such that Σ takes $\partial\Delta^n_\bullet$ to $x_\bullet$, with the latter denoting the image of $\Delta^0_\bullet \to \mathcal{X}$, induced by the vertex inclusion $x \to X$.

More precisely, we have a commutative diagram:

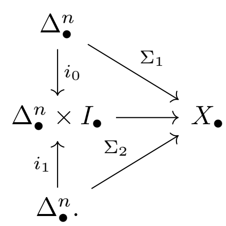
$$\begin{array}{ccc} \Delta^n_\bullet & \longrightarrow & \Delta^0_\bullet \\ & \overset{\Sigma}{\searrow} & \downarrow{\scriptstyle x} \\ & & \mathcal{X}. \end{array}$$

Example A.1. *When $\mathcal{X} = X_\bullet$ is the singular simplicial set of a topological space X, the maps above are in complete correspondence with maps:*

$$\Sigma : \Delta^n \to X,$$

taking the topological boundary of Δ^n to x.

For $X_\bullet$ general simplicial set, a pair of maps $\Sigma_1 : \Delta^n_\bullet \to X_\bullet, \Sigma_2 : \Delta^n_\bullet \to X_\bullet$, are equivalent if there is a diagram, called simplicial homotopy:



such that $\partial \Delta^n_\bullet \times I_\bullet$ is taken by H to $x_\bullet$. The simplicial homotopy groups of a Kan complex $(\mathcal{X}, x)$ coincide with the classical homotopy groups of the geometric realization $(|\mathcal{X}|, x)$.

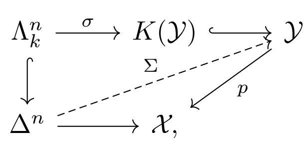
Proof of Lemma 4.2. We refer the reader to Part I, Appendix A.2, for more details on the notions here. We prove a stronger claim.

Lemma A.2. *Let $p : \mathcal{Y} \to \mathcal{X}$ be an inner fibration of quasi-categories $\mathcal{Y}, \mathcal{X}$, with $\mathcal{X}$ a Kan complex. And let $K(\mathcal{Y}) \subset \mathcal{Y}$ denote the maximal Kan subcomplex. Then $p : K(\mathcal{Y}) \to \mathcal{X}$ is a Kan fibration.*

The above is probably well known, but it is simple to just provide the proof for convenience.

Proof. By definition of an inner fibration, whenever we are given a commutative diagram with solid arrows and with $0 < k < n$:

(A.1)



there exists a dashed arrow Σ as indicated, making the whole diagram commutative. When $n > 2$ the edges, i.e. 1-faces, of Σ are all automatically isomorphisms in $\mathcal{Y}$, as Σ extends σ, and all edges of σ are isomorphisms by definition. For $n = 2$ the edges of Σ are either edges of σ, or are compositions of edges of σ in the quasi-category $\mathcal{Y}$, and hence again always invertible.

It follows that Σ maps into $K(\mathcal{Y}) \subset \mathcal{Y}$. Since the starting diagram was arbitrary, we just proved that $p : K(\mathcal{Y}) \to \mathcal{X}$ is an inner fibration. In particular the pre-images $p^{-1}(\Sigma(\Delta^n)) \subset K(\mathcal{Y})$ are quasi-categories, for all n, where $\Sigma : \Delta^n \to \mathcal{X}$ is any n-simplex, see Part I, Appendix A.2. But $K(\mathcal{Y})$ is a Kan complex, so that also the above pre-images $p^{-1}(\Sigma(\Delta^n))$ are Kan complexes. It readily follows from this that $p : K(\mathcal{Y}) \to \mathcal{X}$ is a Kan fibration. □

The main lemma then follows, since if $p : \mathcal{Y} \to \mathcal{X}$ is a Cartesian fibration, it is in particular an inner fibration. □

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