

ELLIPTIC CURVE COUNTS AND GENERALIZED WEINSTEIN CONJECTURES

ABSTRACT. We use elliptic curve Gromov-Witten theory to probe various generalizations of the Weinstein conjectures. For instance, we prove new cases of Mazzucchelli’s generalization of the Weinstein conjecture, and versions of Sandon’s translated fixed points conjecture. We also state an analogue of the Weinstein conjecture in lcs geometry, directly extending the Weinstein conjecture, and discuss various partial verifications. A counterexample for a stronger form of this conjecture is given.

CONTENTS

1. Introduction	1
1.1. Conformal symplectic Weinstein conjecture	3
1.2. Organization	5
2. Preliminaries on <i>lcs</i> on manifolds	5
3. On Reeb 2-curves	7
4. J -holomorphic curves in certain lcs manifolds	8
4.1. Charged elliptic curves in an lcs manifold	10
4.2. Reeb holomorphic tori in $(C \times S^1, J^\lambda)$	11
5. Elements of Gromov-Witten theory of an almost complex manifold	11
6. Elliptic curves in the lcs-ification of a contact manifold and the Fuller index	15
6.1. Preliminaries on Reeb tori	15
7. Proofs of main theorems	19
7.1. Mapping tori and Reeb 2-curves	20
A. Fuller index	25
B. Remark on multiplicity	27
8. Acknowledgements	27
References	27

1. INTRODUCTION

We will start with the more basic story of fixed Reeb strings of strict contactomorphisms. This will turn out to be closely tied to counts of elliptic curves in certain associated lcs manifolds (locally conformally symplectic). From fixed Reeb strings, we are then lead to cases of Mazzucchelli’s generalization of the Weinstein conjecture, Sandon’s translated points, and then to the formulation of the “conformal symplectic Weinstein conjecture”.

Key words and phrases. locally conformally symplectic manifolds, Gromov-Witten theory, virtual fundamental class, Fuller index.

Let (C^{2n+1}, λ) be a compact contact manifold with λ a contact form, that is a one form s.t. $\lambda \wedge (d\lambda)^n \neq 0$. Denote by R^λ the Reeb vector field satisfying:

$$d\lambda(R^\lambda, \cdot) = 0, \quad \lambda(R^\lambda) = 1.$$

Recall that a **closed λ -Reeb orbit** (or just Reeb orbit when λ is implicit) is a smooth map

$$o : (S^1 = \mathbb{R}/\mathbb{Z}) \rightarrow C$$

such that

$$\dot{o}(t) = cR^\lambda(o(t)),$$

with $\dot{o}(t)$ denoting the time derivative, for some $c > 0$ called period. Let $S(R^\lambda, \beta)$ denote the space of all closed Reeb orbits in free homotopy class β , with its compact open topology. And set

$$\mathcal{O}(R^\lambda, \beta) := S(R^\lambda, \beta)/S^1,$$

where S^1 is acting naturally by reparametrization $(\theta \cdot o)(t) = o(t + \theta)$.

Definition 1.1. *Let $\phi : (C, \lambda) \rightarrow (C, \lambda)$ be a strict contactomorphism of a contact manifold. Then a **fixed Reeb string of ϕ in class β** is an element of $o \in \mathcal{O}(R^\lambda, \beta)$ fixed by ϕ .*

Theorem 1.2. *For any contact form λ on S^{2k+1} , sufficiently C^1 nearby to λ_{st} (the standard contact form), any strict contactomorphism ϕ of (C, λ) homotopic to the id via strict contactomorphisms has a fixed Reeb string.*

Let ϕ be strict contactomorphism of (C, λ) . Following Mazzucchelli [12], a **ϕ -invariant Reeb orbit** is a Reeb trajectory γ s.t. for some $T \neq 0$, $\phi(\gamma(t)) = \gamma(t + T)$. Hence *id*-invariant Reeb orbits are precisely closed Reeb orbits. Note also that given a fixed Reeb string of ϕ , represented by γ , the latter is also clearly a ϕ -invariant Reeb orbit.

Conjecture 1 (Mazzucchelli [12]). *For any (C, λ) with C closed, every strict contactomorphism ϕ homotopic to the identity via strict contactomorphisms admits a ϕ -invariant Reeb orbit.*

Theorem 1.2, and more generally Theorem 7.9 give:

Corollary 1.3. *Mazzucchelli's generalization of the Weinstein conjecture [12, Conjecture 1.2], holds for C^∞ perturbations of (S^{2k+1}, λ_{st}) . Furthermore, it holds holds for a definite type contact manifold, see Definition 7.8.*

There is a partial connection of the theorem with the theory of translated points.

Definition 1.4 (Sandon [24]). *Given a general contactomorphism ϕ of (C, λ) , a point $p \in C$ is called a **translated point** provided that $\phi^*\lambda(p) = \lambda(p)$ and $\phi(p)$ lies on the λ -Reeb flow line passing through p .*

A fixed Reeb string for ϕ in particular determines a special translated point of ϕ (one for each point on the image of the fixed Reeb string). Hence we get:

Corollary 1.5. *Let ϕ be a strict contactomorphisms of (S^{2k+1}, λ) for λ sufficiently C^∞ nearby to λ_{st} , then ϕ has a non-empty continuum of translated points.*

Note that the general form of Sandon's conjecture on existence of translated points, [24] has counterexamples on S^{2k+1} for the standard contact form λ_{st} , see Cant [5]. However, it likely holds for any strict contactomorphism of (S^{2k+1}, λ) representing the standard contact structure.

Theorem 1.6. *Suppose that (C, λ) is Morse-Bott and some connected component $N \subset \mathcal{O}(R^\lambda, \beta)$ has non-vanishing Euler characteristic. Then any contact form λ' on C , sufficiently C^1 nearby to λ , any strict contactomorphism ϕ of (C, λ') , homotopic to the id via strict contactomorphisms has a class β fixed Reeb string $o \in N$.*

Both of the theorems above are actually special cases of the following theorem proved in Section 7. For more details on the Fuller index see Appendix A. Let λ be a contact form on a closed manifold C , $N \subset \mathcal{O}(R^\lambda, \beta)$ and let $i(N, R^\lambda, \beta) \in \mathbb{Q}$ denote the Fuller index. For example, if λ is Morse-Bott (see [4]) and N is a connected component of $\mathcal{O}(R^\lambda, \beta)$ then by a computation in [25, Section 2.1.1] $i(R^\lambda, N, \beta) \neq 0$ if $\chi(N) \neq 0$ (the Euler characteristic).

Theorem 1.7. *Let (C, λ) be a contact manifold satisfying the condition: $i(N, R^\lambda, \beta) \neq 0$, for some open compact $N \subset \mathcal{O}(R^\lambda, \beta)$. Then any strict contactomorphism $\phi : (C, \lambda) \rightarrow (C, \lambda)$, homotopic to the id via strict contactomorphisms has a class β fixed Reeb string $o \in N$.*

We have already mentioned that the index assumption of the theorem holds for Morse-Bott contact forms λ , provided the Euler characteristic of some component of $N \subset \mathcal{O}(R^\lambda)$ is non-vanishing. We may take for instance the standard contact form λ_{st} on S^{2k+1} , the unit cotangent bundle of the sphere, or see Bourgeois [4] for more examples. In this Morse-Bott case the theorem may be verified by elementary considerations. To see this, suppose we have a connected component $N \subset \mathcal{O}(R^\lambda)$ with $\chi(N) \neq 0$. Then ϕ as above induces a topological endomorphism $\tilde{\phi}$ of N with non-zero Lefschetz number, so that in this case the result follows by the Lefschetz fixed point theorem.

In general a compact open component $N \subset \mathcal{O}(R^\lambda)$ may not be a finite simplicial complex, or indeed any kind of topological space to which the classical Lefschetz fixed point theorem may apply. Also the relationship of $i(N, R^\lambda, \beta)$ with $\chi(N)$ breaks down in general as $i(N, R^\lambda, \beta)$ is partly sensitive to the dynamics of R^λ .

1.1. Conformal symplectic Weinstein conjecture. The goal is now to describe one analogue of the Weinstein conjecture in locally conformally symplectic geometry, or lcs geometry for short. For preliminaries on this see Section 2.

1.1.1. *Reeb 2-curves.*

Definition 1.8. *Let (M, λ, α) be an exact lcs structure and $\omega = d_\alpha \lambda$. Define X_λ by $\omega(X_\lambda, \cdot) = \lambda$ and X_α by $\omega(X_\alpha, \cdot) = \alpha$. Let \mathcal{D} denote the (generalized) distribution spanned by X_α, X_λ , meaning $\mathcal{D}(p) := \text{span}(X_\alpha(p), X_\lambda(p))$. This will be called the **canonical distribution**.*

The (generalized) distribution \mathcal{D} is one analogue for exact lcs manifolds of the Reeb distribution on contact manifolds. A Reeb 2-curve, as defined ahead, will be a certain kind of singular leaf of \mathcal{D} , and so is a kind of 2-dimensional analogue of a Reeb orbit.

Example 1. The simplest example of a Reeb 2-curve in an exact lcs (M, λ, α) , in the case \mathcal{D} is a true 2-dimensional distribution (for example if (λ, α) is first kind), is a closed immersed surface $u : \Sigma \rightarrow M$ tangent to \mathcal{D} . However, it will be necessary to consider more generalized curves.

Definition 1.9. *Let Σ be a closed nodal Riemann surface (the set of nodes can be empty). Let $u : \Sigma \rightarrow M$ be a smooth map and let $\tilde{u} : \tilde{\Sigma} \rightarrow M$ be its normalization (see Definition 7.1). We say that u is a **Reeb 2-curve** in (M, λ, α) , if the following is satisfied:*

- (1) *For each $z \in \tilde{\Sigma}$, $\tilde{u}_*(T_z \tilde{\Sigma}) = \mathcal{D}(\tilde{u}(z))$, whenever $d\tilde{u}(z) : T_z \tilde{\Sigma} \rightarrow T_{\tilde{u}(z)} M$ is non-zero, and $\dim \mathcal{D}(\tilde{u}(z)) = 2$.*
- (2) $0 \neq [u^* \alpha] \in H^1(\Sigma, \mathbb{R})$.

It is tempting to conjecture that every closed exact lcs manifold has a Reeb 2-curve, in analogy to the Weinstein conjecture. However this is false:

Theorem 1.10. *Let T^2, g_{st} be the 2-torus with its standard flat metric. Let $M_{\tilde{\phi},1}$ be the mapping torus of the unit cotangent bundle of T^2 , with $\tilde{\phi}$ corresponding to an isometry $\phi : (T^2, g_{st}) \rightarrow (T^2, g_{st})$, which does not fix the image of any closed geodesic (an irrational rotation in both coordinates). Then $M_{\tilde{\phi},1}$ has no Reeb 2-curves.*

1.1.2. *Reeb 1-curves.*

Definition 1.11. *A smooth map $o : S^1 \rightarrow M$ is a **Reeb 1-curve** in an exact lcs manifold (M, λ, α) , if*

$$\forall t \in S^1 : (\lambda(o'(t)) > 0) \wedge (o'(t) \in \mathcal{D}).$$

Definition 1.12. *We say that an exact lcs manifold (M, λ, α) satisfies the **Reeb condition** if:*

$$\lambda(X_\alpha) > 0.$$

The following is elementary and is proved in Section 3.

Proposition 1.13. *Every closed exact lcs surface satisfying the Reeb condition has a Reeb 1-curve.*

We will refer to the following as the conformal symplectic Weinstein conjecture.

Conjecture 2. *Suppose that (M, λ, α) is a closed exact lcs manifold of dimension 4 satisfying the Reeb condition, then it has a Reeb 1-curve.*

In special cases, the conjecture reduces to the Weinstein conjecture in dimension three. For example:

Theorem 1.14. *Suppose that (M, λ, α) (as in the statement of the conjecture) has a codimension one closed submanifold C , which is transverse to the Liouville vector field X_λ (defined as above) and such that $d\lambda$ is degenerate along C , then the conjecture is satisfied for this structure.*

Proof. A simple instructive exercise shows that in this case $(C, \lambda|_C)$ is contact. The condition that $d\lambda$ is degenerate along C , readily implies that $\ker d\lambda = \{X_\alpha, X_\lambda\}$ along C , which in turn ensures that any closed $\lambda|_C$ -Reeb orbit, gives a Reeb 1-curve in M . And so by the proof of Taubes [26] of the Weinstein conjecture in dimension three, the conformal symplectic Weinstein conjecture holds in this case. \square

Corollary 3.5 in Section 3 gives another partial verification of the conformal symplectic Weinstein conjecture, for structures nearby to lcs-fications.

Proposition 1.15. *The analogue of Conjecture 2, in all dimensions, implies the Weinstein conjecture: every closed contact manifold (C, λ) has a closed Reeb orbit.*

Proof of Proposition 1.15. Let $(M = C \times S^1, \lambda, \alpha)$ be the lcs-fication of a closed contact manifold (C, λ) . Then it satisfies the Reeb condition. Suppose that $o : S^1 \rightarrow M$ is a Reeb 1-curve. Then

$$\forall t \in [0, 1] : \lambda(\dot{o}(t)) > 0$$

and o is tangent to $\mathcal{V}_\lambda = \mathcal{D}$. Consequently, $pr_C \circ o$ is tangent to $\ker d\lambda$, and

$$\forall \tau \in [0, 1] : \lambda((pr_C \circ o)'(\tau)) > 0.$$

It follows that $pr_C \circ o$ is a Reeb orbit of (C, λ) up to parametrization. \square

1.2. Organization. The main theorems are proved in Section 7. Section 2 presents detailed preliminaries for lcs geometry, which should make this paper self contained and accessible to a general reader.

2. PRELIMINARIES ON lcs ON MANIFOLDS

Definition 2.1. *A locally conformally symplectic manifold or just an lcs manifold, is a smooth $2n$ -fold M with an lcs structure: which is a non-degenerate 2-form ω , with the property that for every $p \in M$ there is an open $U \ni p$ such that $\omega|_U = f_U \cdot \omega_U$, for some symplectic form ω_U defined on U and some smooth positive function f_U on U .*

These kinds of structures were originally considered by Lee in [11], arising naturally as part of an abstract study of “a kind of even dimensional Riemannian geometry”, and then further studied by a number of authors see for instance, [2] and [27]. An lcs manifold admits all the interesting classical notions of a symplectic manifold, like Lagrangian submanifolds and Hamiltonian dynamics, while at the same time forming a much more flexible class. For example Eliashberg and Murphy show that if a closed almost complex $2n$ -fold M has $H^1(M, \mathbb{R}) \neq 0$ then it admits a lcs structure, [7]. Another result of Apostolov, Dloussky [1] is that any complex surface with an odd first Betti number admits a lcs structure, which tames the complex structure.

To see the connection with the first cohomology group $H^1(M, \mathbb{R})$, mentioned above, let us point out right away the most basic invariant of a lcs structure ω , when M has dimension at least 4. This is the Lee class, $\alpha = \alpha_\omega \in H^1(M, \mathbb{R})$. This class has the property that on the associated α -covering space (see proof of Lemma 6.1) \tilde{M} , the lift $\tilde{\omega}$ is globally conformally symplectic. Thus, an lcs form is globally conformally symplectic, that is diffeomorphic to $e^f \cdot \omega'$, with ω' symplectic, iff its Lee class vanishes.

Again assuming M has dimension at least 4, the Lee class α has a natural differential form representative, called the Lee form, which is defined as follows. We take a cover of M by open sets U_a in which $\omega = e^{f_a} \cdot \omega_a$ for ω_a symplectic. Then we have 1-forms $d(f_a)$ on each U_a , which glue to a well-defined closed 1-form on M , as shown by Lee. We may denote this 1-form and its cohomology class both by α . It is moreover immediate that for an lcs form ω ,

$$d\omega = \alpha \wedge \omega,$$

for α the Lee form as defined above.

As we mentioned lcs manifolds can also be understood to generalize contact manifolds. This works as follows. First we have a class of explicit examples of lcs manifolds, obtained by starting with a symplectic cobordism (see [7]) of a closed contact manifold C to itself, arranging for the contact forms at the two ends of the cobordism to be proportional and then gluing the boundary components, (after a global conformal rescaling of the form on the cobordism, to match the boundary conditions).

Terminology 1. For us a contact manifold is a pair (C, λ) where C is a closed manifold and λ a contact form: $\forall p \in C : \lambda \wedge \lambda^{2n}(p) \neq 0$. This is not a completely common terminology as usually it is the equivalence class of (C, λ) that is called a contact manifold, where $(C, \lambda) \sim (C, \lambda')$ if $\lambda = f\lambda'$ for f a positive function. (Given that the contact structure, in the classical sense, is co-oriented.) A **contactomorphism** between $(C_1, \lambda_1), (C_2, \lambda_2)$ is a diffeomorphism $\phi : C_1 \rightarrow C_2$ s.t. $\phi^*\lambda_2 = f\lambda_1$ for some $f > 0$. It is called **strict** if $\phi^*\lambda_2 = \lambda_1$.

A concrete basic example, which can be understood as a special case of the above cobordism construction, is the following.

Example 2 (Banyaga). Let (C, λ) be a contact manifold, $S^1 = \mathbb{R}/\mathbb{Z}$, $d\theta$ the standard non-degenerate 1-form on S^1 satisfying $\int_{S^1} d\theta = 1$. And take $M = C \times S^1$ with the 2-form

$$\omega_\lambda = d_\alpha \lambda := d\lambda - \alpha \wedge \lambda,$$

for $\alpha := pr_{S^1}^* d\theta$, $pr_{S^1} : C \times S^1 \rightarrow S^1$ the projection, and λ likewise the pull-back of λ by the projection $C \times S^1 \rightarrow C$. We call (M, ω_λ) as above the **lcs-fication** of (C, λ) . This is also a basic example of a first kind lcs manifold, as in Definition 2.4 ahead.

The operator

$$(2.2) \quad d_\alpha : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

is called the Lichnerowicz differential with respect to a closed 1-form α , and it satisfies $d_\alpha \circ d_\alpha = 0$ so that we have an associated Lichnerowicz chain complex.

Definition 2.3. *An exact lcs form on M is an lcs 2-form s.t. there exists a pair of one forms (λ, α) with α a closed 1-form, s.t. $\omega = d_\alpha \lambda$ is non-degenerate. In the case above we also call the pair (λ, α) an exact lcs structure. The triple (M, λ, α) will be called an exact lcs manifold, but we may also call (M, ω) an exact lcs manifold when (λ, α) are implicit.*

An exact lcs structure determines a generalized distribution \mathcal{V}_λ on M :

$$\mathcal{V}_\lambda(p) = \{v \in T_p M \mid d\lambda(v, \cdot) = 0\},$$

which we call the **vanishing distribution**. We also define a generalized distribution ξ_λ that is the ω -orthogonal complement to \mathcal{V}_λ , which we call **co-vanishing distribution**. For each $p \in M$, $\mathcal{V}_\lambda(p)$ has dimension at most 2 since $d\lambda - \alpha \wedge \lambda$ is non-degenerate. If M^{2n} is closed \mathcal{V}_λ cannot identically vanish since $(d\lambda)^n$ cannot be non-degenerate by Stokes theorem.

Definition 2.4. *Let (λ, α) be an exact lcs structure on M . We call α integral, rational or irrational if its periods are integral, respectively rational, or respectively irrational. We call the structure (λ, α) scale integral, if $c\alpha$ is integral for some $0 \neq c \in \mathbb{R}$. Otherwise we call the structure scale irrational. If \mathcal{V} is non-zero at each point of M , in particular is a smooth 2-distribution, then such a structure is called first kind. If ω is an exact lcs form then we call ω integral, rational, irrational, first kind if there exists λ, α s.t. $\omega = d_\alpha \lambda$ and (λ, α) is integral, respectively irrational, respectively first kind. Similarly define, scale integral, scale irrational ω .*

Definition 2.5. *A conformal symplectomorphism of lcs manifolds $\phi : (M_1, \omega_1) \rightarrow (M_2, \omega_2)$ is a diffeomorphism ϕ s.t. $\phi^* \omega_2 = e^f \omega_1$, for some f . Note that in this case we have an induced relation (when M has dimension at least 4):*

$$\phi^* \alpha_1 = \alpha_0 + df,$$

where α_1 is the Lee form of ω_1 and α_0 is the Lee form of ω_0 . If $f = 0$ we call ϕ a **symplectomorphism**. A (conformal) symplectomorphism of exact lcs structures (λ_1, α_1) , (λ_2, α_2) on M_1 respectively M_2 is a (conformal) symplectomorphism of the corresponding lcs 2-forms. If a diffeomorphism $\phi : M_1 \rightarrow M_2$ satisfies $\phi^* \lambda_2 = \lambda_1$ and $\phi^* \alpha_2 = \alpha_1$ we call it an **isomorphism** of the exact lcs structures. This is analogous to a strict contactomorphism of contact manifolds.

To summarize, with the above notions we have the following basic points whose proof is left to the reader:

- (1) An isomorphism of exact lcs structures (λ_1, α_1) , (λ_2, α_2) preserves the first kind condition, and moreover preserves the corresponding vanishing distributions.
- (2) A symplectomorphism of lcs forms preserves the first kind condition.

- (3) A conformal symplectomorphism of exact lcs structures generally does not preserve the first kind condition.
- (4) A (conformal) symplectomorphism of first kind lcs structures generally does not preserve the vanishing distributions. (Similar to a contactomorphism not preserving Reeb distributions.)
- (5) A conformal symplectomorphism of lcs forms and exact lcs structures preserves the rationality, integrality, scale integrality conditions.

Remark 2.6. In view of the above, in the context of the paper, especially in the context of the conformal symplectic Weinstein conjecture, the correct “gauge group” is the group of symplectomorphisms, rather than conformal symplectomorphisms. Although neither gauge group will be studied in this note.

Remark 2.7. We say that ω_0 is conformally equivalent to ω_1 if $\omega_1 = e^f \omega_0$, i.e. the identity map is a conformal symplectomorphism $id : (M, \omega_0) \rightarrow (M, \omega_1)$. It is important to note that for us the form ω is the structure not its conformal equivalence class, as for some authors. In other words conformally equivalent structures on a given manifold determine distinct but isomorphic objects of the category, whose objects are lcs manifolds and morphisms conformal symplectomorphisms.

Example 3. One example of an lcs structure of the first kind is a mapping torus of a strict contactomorphism, see Banyaga [2]. The mapping tori $M_{\phi, c}$ of a strict contactomorphism ϕ of (C, λ) fiber over S^1 ,

$$\pi : C \hookrightarrow M_{\phi, c} \rightarrow S^1,$$

with Lee form of the type $\alpha = c\pi^*(d\theta)$, for some $0 \neq c \in \mathbb{R}$. In particular, these are scale integral first kind lcs structures.

Moreover we have:

Theorem 2.8 (Only reformulating Bazzoni-Marrero [3]). *A first kind lcs structure (λ, α) on a closed manifold M is isomorphic to the mapping torus of a strict contactomorphism if and only if it is scale integral.*

The (scale) integrality condition is of course necessary since the Lee form of a mapping torus of a strict contactomorphism will have this property. Thus we may understand scale irrational first kind lcs structures as first (and rather dramatic) departures from the world of contact manifolds into a brave new lcs world.

Remark 2.9. Note that scale irrational first kind structures certainly exist. A simple example is given by taking λ, α to be closed scale irrational 1-forms on T^2 with transverse kernels. Then $\omega = \lambda \wedge \alpha$ is a scale irrational first kind structure on T^2 . In particular (λ, α) cannot be a mapping torus of a strict contactomorphism even up to a conformal symplectomorphism. In general, on a closed manifold we may always perturb a (first kind) scale integral lcs structure to a (first kind) scale irrational one. The examples of the present paper deal with deformations of this sort.

3. ON REEB 2-CURVES

Definition 3.1. *Define the set $\mathcal{L}(M)$ of exact lcs structures on M , to be:*

$$\mathcal{L}(M) = \{(\beta, \gamma) \in \Omega^1(M) \times \Omega^1(M) \mid \gamma \text{ is closed, } d_\gamma \beta \text{ is non-degenerate}\}.$$

Define $\mathcal{F}(M) \subset \mathcal{L}(M)$ to be subset of (possibly irrational) first kind lcs structures.

In what follows we use the following C^∞ metric on $\mathcal{L}(M)$. For $(\lambda_1, \alpha_1), (\lambda_2, \alpha_2) \in \mathcal{L}(M)$ define:

$$(3.2) \quad d_\infty((\lambda_1, \alpha_1), (\lambda_2, \alpha_2)) = d_{C^\infty}(\lambda_1, \lambda_2) + d_{C^\infty}(\alpha_1, \alpha_2),$$

where d_{C^∞} on the right side is the usual C^∞ metric.

The following theorems are proved in Section 7, based on the theory of elliptic pseudo-holomorphic curves in M .

Theorem 3.3. *Let (C, λ) be a closed contact manifold, satisfying one of the following conditions:*

- (1) (C, λ) has at least one non-degenerate Reeb orbit.
- (2) $i(N, R^\lambda, \beta) \neq 0$ where the latter is the Fuller index of some open compact subset of the orbit space: $N \subset \mathcal{O}(R^\lambda, \beta)$, as previously.

Then we have the following:

- (1) For some d_∞ neighborhood U of the lcs-fication (λ, α) of the space $\mathcal{F}(M = C \times S^1)$, every element of U admits a Reeb 2-curve.
- (2) For any $(\lambda', \alpha') \in U$, the corresponding Reeb 2-curve $u : \Sigma \rightarrow M$ can be assumed to be **elliptic** meaning that Σ is elliptic (more specifically: a nodal, topological genus 1, closed, connected Riemann surface).
- (3) u can also be assumed to be α -charge 1 (see Definition 4.3).
- (4) If M has dimension 4 then u can be assumed to be embedded and normal (the set of nodes is empty). And so in particular, such a u represents a closed, $(\omega = d_\alpha \lambda)$ -symplectic torus hypersurface.

Theorem 3.4. *Suppose that (M, λ, α) is an exact lcs manifold satisfying the Reeb condition. If (M, λ, α) has an immersed Reeb 2-curve then it also has a Reeb 1-curve. Furthermore, if it has an immersed elliptic Reeb 2-curve, then this Reeb 2-curve is normal.*

We have an immediate corollary of Theorem 3.3 and Theorem 3.4.

Corollary 3.5. *Let λ be a contact form, on closed 3-manifold C , with at least one non-degenerate Reeb orbit, or more generally satisfying $i(N, R^\lambda, \beta) \neq 0$ for some open compact N as previously. Then there is a d_∞ neighborhood U of the lcs-fication (λ, α) in the space $\mathcal{F}(C \times S^1)$, s.t. for each $(\lambda', \alpha') \in U$ there is a Reeb 1-curve.*

Lemma 3.6. *Let (M, λ, α) be an exact lcs manifold with M closed then $0 \neq [\alpha] \in H^1(M, \mathbb{R})$.*

Proof. Suppose by contradiction that α is exact and let g be its primitive. Then computing we get: $d_\alpha \lambda = \frac{1}{f} d(f\lambda)$ with $f = e^g$. Consequently, $d(f\lambda)$ is non-degenerate on M which contradicts Stokes theorem. \square

Proof of Proposition 1.13. This follows by Theorem 3.4. As in this case by the lemma above the identity map $X \rightarrow X$ is an immersed Reeb 2-curve, by Lemma 3.6. \square

4. J -HOLOMORPHIC CURVES IN CERTAIN LCS MANIFOLDS

Let M, J be an almost complex manifold and Σ, j a Riemann surface. Recall that a map $u : \Sigma \rightarrow M$ is said to be J -holomorphic if $du \circ j = J \circ du$.

Notation 1. We will often say J -**curve** in place of J -holomorphic curve.

First kind lcs manifolds give immediate examples of almost complex manifolds where the L^2 energy functional is unbounded on the moduli spaces of fixed homology class J -curves, as well as where null-homologous J -curves can be non-constant. We are going to see this shortly after developing a more general theory.

Definition 4.1. *Let (M, λ, α) be an exact lcs manifold, satisfying the **Reeb condition**: $\omega(X_\lambda, X_\alpha) = \lambda(X_\alpha) > 0$, where $\omega = d_\alpha \lambda$. In this case, \mathcal{D} is a 2-dimensional distribution, and we say that an ω -compatible J is (λ, α) -**admissible** or ω -admissible (when λ, α are implicit) if:*

- J preserves the canonical distribution \mathcal{D} and preserves the ω -orthogonal complement \mathcal{D}^\perp of \mathcal{D} . That is $J(V) \subset \mathcal{D}$ and $J(\mathcal{D}^\perp) \subset \mathcal{D}^\perp$.
- $d\lambda$ tames J on \mathcal{D}^\perp .

Admissible J exist by classical symplectic geometry, and the space of such J is contractible see [17]. We call (λ, α, J) as above a **tamed exact lcs structure**, and (ω, J) is called a *tamed exact lcs structure* if $\omega = d_\alpha \lambda$, for (λ, α, J) a tamed exact lcs structure. In this case (M, ω, J) , (M, λ, α, J) will be called a **tamed exact lcs manifold**.

Example 4. If (M, λ, α) is first kind then by an elementary computation $\omega(X_\lambda, X_\alpha) = 1$ everywhere. In particular, we may find a J such that (λ, α, J) is a tamed exact lcs structure, and the space of such J is contractible. We will call (M, λ, α, J) a **tamed first kind** lcs manifold.

Lemma 4.2. *Let (M, λ, α, J) be a tamed first kind lcs manifold. Then given a smooth $u : \Sigma \rightarrow M$, where Σ is a closed (nodal) Riemann surface, u is J -holomorphic only if*

$$\text{image } d\tilde{u}(z) \subset \mathcal{V}_\lambda(\tilde{u}(z))$$

for all $z \in \tilde{\Sigma}$, where $\tilde{u} : \tilde{\Sigma} \rightarrow M$ is the normalization of u (see Definition 7.1). In particular $\tilde{u}^* d\lambda = 0$.

Proof. As previously observed, by the first kind condition, \mathcal{V}_λ is the span of X_λ, X_α and hence

$$V := \mathcal{V}_\lambda = \mathcal{D}_\lambda.$$

Let u be J -holomorphic, so that \tilde{u} is J -holomorphic (by definition of a J -holomorphic nodal map). We have

$$\int_{\tilde{\Sigma}} \tilde{u}^* d\lambda = 0$$

by Stokes theorem. Let $\text{proj}(p) : T_p M \rightarrow V^\perp(p)$ be the projection induced by the splitting $TM = V \oplus V^\perp$.

Suppose that for some $z \in \tilde{\Sigma}$, $\text{proj} \circ d\tilde{u}(z) \neq 0$. By the conditions:

- J is tamed by $d\lambda$ on V^\perp .
- $d\lambda$ vanishes on V .
- J preserves the splitting $TM = V \oplus V^\perp$.

we have $\int_{\tilde{\Sigma}} \tilde{u}^* d\lambda > 0$, a contradiction. Thus,

$$\forall z \in \tilde{\Sigma} : \text{proj} \circ d\tilde{u}(z) = 0,$$

so

$$\forall z \in \tilde{\Sigma} : \text{image } d\tilde{u}(z) \subset \mathcal{V}_\lambda(\tilde{u}(z)).$$

□

Example 5. Let $(C \times S^1, \lambda, \alpha)$ be the lcs-fication of a contact manifold (C, λ) . In this case

$$X_\alpha = (R^\lambda, 0),$$

where R^λ is the Reeb vector field and

$$X_\lambda = \left(0, \frac{d}{d\theta}\right)$$

is the vector field generating the natural action of S^1 on $C \times S^1$.

If we denote by $\xi \subset T(C \times S^1)$ the distribution $\xi(p) = \ker \lambda(p)$, then in this case $\xi = V^\perp$ in the notation above. We then take J to be an almost complex structure on ξ , which is S^1 invariant, and compatible with $d\lambda$. The latter means that

$$g_J(\cdot, \cdot) := d\lambda|_{\xi}(\cdot, J\cdot)$$

is a J invariant Riemannian metric on the distribution ξ .

There is an induced almost complex structure J^λ on $C \times S^1$, which is S^1 -invariant, coincides with J on ξ and which satisfies:

$$J^\lambda(X_\alpha) = X_\lambda.$$

Then $(C \times S^1, \lambda, \alpha, J^\lambda)$ is a tamed integral first kind lcs manifold.

4.1. Charged elliptic curves in an lcs manifold. We now study moduli spaces of elliptic curves in a lcs manifold, constrained to have a certain charge.¹ In the present context, one reason for the introduction of “charge” is that it is now possible for non-constant holomorphic curves to be null-homologous, so we need additional control. Here is a simple example: take $S^3 \times S^1$ with $J = J^\lambda$, for the λ the standard contact form, then all the Reeb holomorphic tori (as defined further below) are null-homologous.

Let Σ be a complex torus with a chosen marked point $z \in \Sigma$, i.e. an elliptic curve over \mathbb{C} . An isomorphism $\phi : (\Sigma_1, z_1) \rightarrow (\Sigma_2, z_2)$ is a biholomorphism s.t. $\phi(z_1) = z_2$. The set of isomorphism classes forms a smooth orbifold $M_{1,1}$. This has a natural compactification - the Deligne-Mumford compactification $\overline{M}_{1,1}$, by adding a point at infinity, corresponding to a nodal genus 1 curve with one node.

The notion of charge can be defined in a general setting. Let M be a manifold endowed with a closed integral 1-form α . Let $u : T^2 \rightarrow M$ be a continuous map satisfying $[u^*\alpha] \neq 0 \in H^1(T^2, \mathbb{Z})$. We then have a classifying map $g : T^2 \rightarrow S^1$ s.t. $g^*d\theta = c \cdot u^*\alpha$ for some c . Take a component of a regular fiber of g , and let $[\gamma] \in H_1(T^2, \mathbb{Z})$ denote the corresponding class, represented by $\gamma : S^1 \rightarrow T^2$. Let $[\rho] \in H_1(T^2, \mathbb{Z})$, likewise represented by $\rho : S^1 \rightarrow T^2$, be the Poincare dual class so that:

$$\langle [\gamma], u^*\alpha \rangle = 0, \quad \langle [\rho], u^*\alpha \rangle \neq 0,$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing of homology and cohomology. These $[\gamma], [\rho]$ generate $H_1(T^2, \mathbb{Z})$, and $[\gamma] \cdot [\rho] = 1$, where \cdot is the intersection pairing with respect to the standard complex orientation on T^2 .

Definition 4.3. For u, γ, ρ as above. We call

$$n = |\langle [\rho], u^*\alpha \rangle| \in \mathbb{N}_{>0},$$

the α -charge of u , or just the charge of u when α is implicit. Suppose furthermore that $\langle \rho, u^*\alpha \rangle > 0$. Then the class $[u \circ \gamma] \in \pi_1(M)$ will be called the π -class of u , for $\pi_1(M)$ the set of free homotopy classes of loops as before.

¹The name charge is inspired by the notion of charge in Oh-Wang [21], in the context of contact instantons. However, the respective notions are not obviously related.

It is easy to see that the α -charge and π -class is independent of the choices above. We may extend the definition of charge to curves $u : \Sigma \rightarrow M$, with Σ a nodal elliptic curve, as follows. If $\rho : S^1 \rightarrow \Sigma$ represents the generator of $H_1(\Sigma, \mathbb{Z})$ then define the charge of u to be $|\langle \rho, u^* \alpha \rangle|$. Obviously the charge condition is preserved under Gromov convergence of stable maps. But it is not preserved in homology, so that charge is *not* a functional $H_2(M, \mathbb{Z}) \rightarrow \mathbb{N}$.

Definition 4.4. *By the above, associated to a continuous map $u : \Sigma \rightarrow M$ with Σ an elliptic curve, and non-zero α -charge, we have a triple $(A, \beta, n) \in H^2(M, \mathbb{Z}) \times \pi_1(M) \times \mathbb{N}_{>0}$, corresponding to the homology class, the π -class, and the α -charge. This triple will be called the **charge class** of u .*

Let (M, J) be an almost complex manifold and α a closed integral 1-form on M non vanishing in cohomology, then we call (M, J, α) a **Lee manifold**. Suppose for the moment that there are no non-constant J -holomorphic maps $(S^2, j) \rightarrow (M, J)$ (otherwise we need stable maps), then for $n \geq 1$ we define:

$$\overline{\mathcal{M}}_{1,1}^n(J, A, \beta)$$

as the set of equivalence classes of tuples (u, S) , for $S = (\Sigma, z)$ a possibly nodal elliptic curve and $u : \Sigma \rightarrow M$ a charge class (A, β, n) , J -holomorphic map. The equivalence relation is $(u_1, S_1) \sim (u_2, S_2)$ if there is an isomorphism $\phi : S_1 \rightarrow S_2$ s.t. $u_2 \circ \phi = u_1$. It is not hard to see that such an isomorphism of preserves the charge class, so that $\overline{\mathcal{M}}_{1,1}^n(J, A, \beta)$ is well defined.

Also note that the expected dimension of $\overline{\mathcal{M}}_{1,1}^1(J^\lambda, A, \beta)$ is 0. It is given by the Fredholm index of the operator (6.5) which is 2, minus the dimension of the reparametrization group (for non-nodal curves) which is 2. That is given an elliptic curve $S = (\Sigma, z)$, let $\mathcal{G}(\Sigma)$ be the 2-dimensional group of biholomorphisms ϕ of Σ . Then given a J -holomorphic map $u : \Sigma \rightarrow M$, (Σ, z, u) is equivalent to $(\Sigma, \phi(z), u \circ \phi)$ in $\overline{\mathcal{M}}_{1,1}^1(J^\lambda, A, \beta)$, for $\phi \in \mathcal{G}(\Sigma)$.

By slight abuse we may just denote such an equivalence class above simply by u , so we may write $u \in \overline{\mathcal{M}}_{1,1}^n(J, A, \beta)$, with S implicit.

4.2. Reeb holomorphic tori in $(C \times S^1, J^\lambda)$. In this section we discuss an important example. Let (C, λ) be a contact manifold and let α and J^λ be as in Example 5. So that in particular we get a Lee manifold $(C \times S^1, J^\lambda, \alpha)$.

In this case we have one natural type of charge 1 J^λ -holomorphic tori in $M = C \times S^1$. Let o be a period c , closed Reeb orbit o of R^λ , and let β it's class in $\pi_1(C) \subset \pi_1(M)$. A **Reeb torus** u_o for o is the map

$$\begin{aligned} u_o : (S^1 \times S^1 = T^2) &\rightarrow C \times S^1 \\ u_o(s, t) &= (o(s), t). \end{aligned}$$

A Reeb torus is J^λ -holomorphic for a uniquely determined holomorphic structure j on T^2 defined by:

$$j\left(\frac{\partial}{\partial s}\right) = c \frac{\partial}{\partial t}.$$

5. ELEMENTS OF GROMOV-WITTEN THEORY OF AN ALMOST COMPLEX MANIFOLD

Suppose that (M, J) is an almost complex manifold (possibly non-compact), where the almost complex structures J are assumed throughout the paper to be C^∞ . Let $N \subset \overline{\mathcal{M}}_{g,k}(J, A)$ be an open compact subset (not necessarily connected) with energy positive on N . The latter energy condition is only relevant when $A = 0$. We shall primarily refer in what follows to work of Pardon in [22], being more familiar to the author. However, the original setup of Fukaya-Ono [10], and later expanded with Oh-Ohta [9] can also work here in the same fashion.

The construction in [22] of an implicit atlas, on the moduli space \mathcal{M} of J -curves (possibly nodal with homological and or other constraints) ostensibly asks for J to be tamed by a symplectic form. But this is only needed to establish compactness. More specifically, suppose we have a general almost complex pair (M, J) and a subset $N \subset \mathcal{M}$ that is open and compact. Then the construction in [22] works word for word to establish existence of a natural implicit atlas on N . The openness condition is needed since the implicit atlas construction needs neighborhoods of elements in \mathcal{M} in the space of all curves (likewise for Kuranishi structures.)

So for an *open* compact component N as above, we have a well defined natural implicit atlas, (or a Kuranishi structure in the setup of [10]). And so such an N will have a virtual fundamental class in the sense of [22]. This understanding will be used in other parts of the paper, following Pardon for the explicit setup.

We may thus define functionals:

$$(5.1) \quad GW_{g,n}(N, J, A) : H_*(\overline{\mathcal{M}}_{g,n}) \otimes H_*(M) \rightarrow \mathbb{Q}.$$

In our more specific context we must in addition restrict the charge, which is defined at the moment for genus 1 curves. So supposing (M, J, α) is a Lee manifold we may likewise define functionals:

$$(5.2) \quad GW_{1,1}^k(N, J, A, \beta) : H_*(\overline{\mathcal{M}}_{1,1}) \otimes H_*(M) \rightarrow \mathbb{Q},$$

meaning that we restrict the count to charge class (A, β, k) curves, with $N \subset \overline{\mathcal{M}}_{1,1}^k(J, A, \beta)$, an open compact subset. If N is not specified it is understood to be the whole moduli space (if it is known to be compact).

We now study how functionals depend on N, J . To avoid unnecessary generality, we discuss the case of $GW_{1,1}^k(N, J, A, \beta)$. Given a Frechet smooth family $\{J_t\}$, $t \in [0, 1]$, on M , we denote by $\overline{\mathcal{M}}_{1,1}^k(\{J_t\}, A, \beta)$ the space of pairs (u, t) , $u \in \overline{\mathcal{M}}_{1,1}^k(J_t, A, \beta)$.

Lemma 5.3. *Let $\{J_t\}$, $t \in [0, 1]$ be a Frechet smooth family of almost complex structures on M . Suppose that \tilde{N} is an open compact subset of the cobordism moduli space $\overline{\mathcal{M}}_{1,1}^k(\{J_t\}, A, \beta)$, with $k > 0$. Let*

$$N_i = \tilde{N} \cap \left(\overline{\mathcal{M}}_{1,1}^k(J_i, A, \beta) \right),$$

then

$$GW_{1,1}^k(N_0, J_0, A) = GW_{1,1}^k(N_1, J_1, A, \beta).$$

In particular if $GW_{1,1}^k(N_0, A, J_0, \beta) \neq 0$, there is a J_1 -holomorphic, stable, charge class (A, β, k) elliptic curve in M .

Proof of Lemma 5.3. We may construct exactly as in [22] a natural implicit atlas on \tilde{N} , with boundary $N_0^{op} \sqcup N_1$, (*op* denoting opposite orientation). And so we immediately get

$$GW_{1,1}^k(N_0, J_0, A, \beta) = GW_{1,1}^k(N_1, J_1, A, \beta).$$

□

Remark 5.4. The condition that $k > 0$ is a simple way to rule out degenerations to constant curves, but is not really essential. In the case the manifold is closed, degenerations of J -holomorphic curves to constant curves are impossible. This can be deduced from energy quantization coming from the general monotonicity theorem as appearing in Zinger [31, Proposition 3.12]. This was noted to me by Spencer Cattalani. Even if the manifold is not compact, given the assumption that \tilde{N} itself is compact, we may similarly preclude such degenerations.

We now state a basic technical lemma, following some standard definitions.

Definition 5.5. An **almost symplectic pair** on M is a tuple (ω, J) , where ω is a non-degenerate 2-form on M , and J is ω -compatible, meaning that $\omega(\cdot, J\cdot)$ defines J -invariant Riemannian metric, denoted by g_J (with ω implicit).

Definition 5.6. We say that a pair of almost symplectic pairs (ω_i, J_i) are δ -**close**, if ω_0, ω_1 are C^∞ δ -close, and J_0, J_1 are C^∞ δ -close, $i = 0, 1$.

Let $\mathcal{S}(A)$ denote the space of equivalence classes of all smooth, nodal, stable, charge k , elliptic curves in M in class A , with the standard Gromov topology determined by g_J . That is elements of $\mathcal{S}(A)$ are like elements of $\overline{\mathcal{M}}_{1,1}^k(J, A, \beta)$ but are not required to be J -holomorphic. In particular, we have a continuous function:

$$e = e_{g_J} : \mathcal{S}(A) \rightarrow \mathbb{R}_{\geq 0}.$$

Lemma 5.7. Let (ω, J) be an almost symplectic pair on a compact manifold M and let $N \subset \overline{\mathcal{M}}_{1,1}^k(J, A, \beta)$ be compact and open (as a subset of $\overline{\mathcal{M}}_{1,1}^k(J, A)$). Then there exists an open $U \subset \mathcal{S}(A)$ satisfying:

- (1) e is bounded on \overline{U} .
- (2) $U \supset N$.
- (3) $\overline{U} \cap \overline{\mathcal{M}}_{1,1}^k(J, A, \beta) = N$.

Proof. The Gromov topology on $\mathcal{S}(A)$ has a basis \mathcal{B} satisfying:

- (1) If $V \in \mathcal{B}$ then e is bounded on \overline{V} .
- (2) If U is open and $u \in U$, then

$$\exists V \in \mathcal{B} : (u \in V) \wedge (\overline{V} \subset U).$$

In the genus 0 case this is contained in the classical text McDuff-Salamon [18, page 140]. The basis \mathcal{B} is defined using a collection of “quasi distance functions” $\{\rho_\epsilon\}_\epsilon$ on the set stable maps. The higher genus case is likewise well known.

Thus, since N is relatively open, using the properties of \mathcal{B} above, we may find a collection $\{V_\alpha\} \subset \mathcal{B}$ s.t.

- $\{V_\alpha\}$ covers N .
- $\overline{V}_\alpha \cap \overline{\mathcal{M}}_{1,1}^k(J, A, \beta) \subset N$.

As N is compact, we have a finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$. Set $U := \cup_{i \in \{1, \dots, n\}} V_{\alpha_i}$. Then U satisfies the conclusion of the lemma. \square

Lemma 5.8. Let (M, ω, J, α) be as above, $N \subset \overline{\mathcal{M}}_{1,1}^k(J, A, \beta)$ an open compact set, and U as in the lemma above. Then there is a $\delta > 0$ s.t. whenever J' is C^2 δ -close to J if $u \in \overline{\mathcal{M}}_{1,1}^k(J', A, \beta)$ and $u \in \overline{U}$ then $u \in U$.

Proof. Suppose otherwise, then there is a sequence $\{J_k\}$ C^2 converging to J , and a sequence $\{u_k\} \in \overline{U} - U$ of J_k -holomorphic stable maps. Then by property 1 e_{g_J} is bounded on $\{u_k\}$. Hence, by Gromov compactness, specifically theorems [18, B.41, B.42], we may find a Gromov convergent subsequence $\{u_{k_j}\}$ to a J -holomorphic stable map $u \in \overline{U} - U$. But by Properties 3, 2 of the set U ,

$$(\overline{U} - U) \cap \overline{\mathcal{M}}_{1,1}^k(J, A, \beta) = \emptyset.$$

So that we obtain a contradiction. \square

Lemma 5.9. *Let M, ω, J, α and $N \subset \overline{\mathcal{M}}_{1,1}^k(J, A, \beta)$ be as in the previous lemma. Then there is a $\delta > 0$ s.t. the following is satisfied. Let (ω', J') be δ -close to (ω, J) , then there is a continuous in the C^∞ topology family $\{J_t\}$, $J_0 = J$, $J_1 = J'$ s.t. there is an open compact subset*

$$\tilde{N} \subset \overline{\mathcal{M}}_{1,1}^k(\{J_t\}, A, \beta),$$

satisfying

$$\tilde{N} \cap \overline{\mathcal{M}}_{1,1}^k(J, A, \beta) = N.$$

Proof. First let δ be as in Lemma 5.8. We then need:

Lemma 5.10. *Given a $\delta > 0$ there is a $\delta' > 0$ s.t. if (ω', J') is δ' -near (ω, J) then there is a continuous in the C^∞ topology family $\{(\omega_t, J_t)\}$ satisfying:*

- (ω_t, J_t) is δ -close to (ω, J) for each t .
- $(\omega_0, J_0) = (\omega, J)$ and $(\omega_1, J_1) = (\omega', J')$.

Proof. Let $\{g_t\}$ be the family of metrics on M given by the convex linear combination of $g = g_{\omega, J}$, $g' = g_{\omega', J'}$, $g_t = (1-t)g + tg'$. Clearly g_t is C^∞ δ' -close to g_0 for each t . Likewise, the family of 2 forms $\{\omega_t\}$ given by the convex linear combination of ω, ω' is non-degenerate for each t if δ' was chosen to be sufficiently small. And each ω_t is C^∞ δ' -close to $\omega_0 = \omega_{g, J}$.

Let

$$ret : Met(M) \times \Omega(M) \rightarrow \mathcal{J}(M)$$

be the ‘‘retraction map’’ (it can be understood as a retraction followed by projection) as defined in [17, Prop 2.50], where $Met(M)$ is space of metrics on M , $\Omega(M)$ the space of 2-forms on M , and $\mathcal{J}(M)$ the space of almost complex structures. This map has the property that the almost complex structure $ret(g, \omega)$ is compatible with ω , and that $ret(g_J, \omega) = J$ for $g_J = \omega(\cdot, J\cdot)$. Then $\{(\omega_t, ret(g_t, \omega_t))\}$ is a compatible family. As ret is continuous in C^∞ -topology, δ' can be chosen such that $\{ret_t(g_t, \omega_t)\}$ are C^∞ δ -nearby. \square

Returning to the proof of the main lemma. Let $\delta' < \delta$ be chosen as in Lemma 5.10 and let $\{(\omega_t, J_t)\}$ be the corresponding family. Set

$$\tilde{N} = \overline{\mathcal{M}}_{1,1}^k(\{J_t\}, A, \beta) \cap (U \times [0, 1]),$$

where U is as in Lemma 5.8.

Then \tilde{N} is an open subset of $\overline{\mathcal{M}}_{1,1}^k(\{J_t\}, A, \beta)$. By Lemma 5.8,

$$\tilde{N} = \overline{\mathcal{M}}_{1,1}^k(\{J_t\}, A, \beta) \cap (\overline{U} \times [0, 1]),$$

so that \tilde{N} is also closed.

Finally, $\sup_{(u,t) \in \tilde{N}} e_{g_t}(u) < \infty$, by condition 1 of U , and since $\{e_{g_t}\}$, $t \in [0, 1]$ is a continuous family. Consequently \tilde{N} is compact by the Gromov compactness theorem. Resetting $\delta := \delta'$, we are then done with the proof of the main lemma. \square

Proposition 5.11. *Given an almost complex manifold M, J suppose that $N \subset \overline{\mathcal{M}}_{1,1}^k(J, A)$ is open and compact. Suppose also that $GW_{1,1}^k(N, J, A, \beta) \neq 0$. Then there is a $\delta > 0$ s.t. whenever J' is C^2 δ -close to J , there exists $u \in \overline{\mathcal{M}}_{1,1}^k(J', A, \beta)$.*

Proof. For N as in the hypothesis, let U , δ and \tilde{N} be as in Lemma 5.9, then by the conclusion of that lemma and by Lemma 5.3

$$GW_{1,1}^k(N_1, J', A, \beta) = GW_{1,1}^k(N, J, A, \beta) \neq 0,$$

where $N_1 = \tilde{N} \cap \overline{\mathcal{M}}_{1,1}^k(J_1, A, \beta)$. \square

6. ELLIPTIC CURVES IN THE LCS-FICATION OF A CONTACT MANIFOLD AND THE FULLER INDEX

The following elementary result is crucial for us.

Lemma 6.1. *Let (M, λ, α, J) be a tamed first kind lcs manifold. Then every non-constant (nodal) J -holomorphic curve $u : \Sigma \rightarrow M$ is a Reeb 2-curve.*

Proof of Lemma 6.1. Let $u : \Sigma \rightarrow M$ be a non-constant, nodal (possibly empty set of nodes) J -curve. By Lemma 4.2 it is enough to show that $[u^*\alpha] \neq 0$. Let \tilde{M} denote the α -covering space of M , that is the space of equivalence classes of paths p starting at $x_0 \in M$, with a pair p_1, p_2 equivalent if $p_1(1) = p_2(1)$ and

$$\int_{[0,1]} p_1^* \alpha = \int_{[0,1]} p_2^* \alpha.$$

Then the lift of ω to \tilde{M} is

$$\tilde{\omega} = \frac{1}{f} d(f\lambda),$$

where $f = e^{-g}$ and where g is a primitive for the lift $\tilde{\alpha}$ of α to \tilde{M} , that is $\tilde{\alpha} = dg$. In particular $\tilde{\omega}$ is conformally symplectomorphic to an exact symplectic form on \tilde{M} . So if \tilde{J} denotes the lift of J , any closed \tilde{J} -curve is constant by Stokes theorem. Now if $[u^*\alpha] = 0$ then u has a lift to a \tilde{J} -holomorphic map $v : \Sigma \rightarrow \tilde{M}$. Since Σ is closed, it follows by the above that v is constant, so that u is constant, which is impossible. \square

6.1. Preliminaries on Reeb tori. Let $(M = C \times S^1, \lambda, \alpha)$ be the lcs-fication of (C, λ) . For $\beta \in \pi_1(C)$ we set $A_\beta^1 = \beta \otimes [S^1] \in H_2(M, \mathbb{Z})$. Let $\mathcal{O}(R^\lambda, \beta)$, be the orbit space as in the Introduction. Let J^λ on $C \times S^1$ be as in Section 4.2. We have a map:

$$(6.2) \quad \mathcal{P} : \mathcal{O}(R^\lambda, \beta) \rightarrow \overline{\mathcal{M}}_{1,1}^1(J^\lambda, A_\beta^1, \beta), \quad \mathcal{P}(o) = u_o,$$

for u_o the Reeb torus as previously. We can say more:

Proposition 6.3. *For any (λ, α) -admissible J there is a natural bijection:*²

$$\mathcal{P} : \mathcal{O}(R^\lambda, \beta) \rightarrow \overline{\mathcal{M}}_{1,1}^1(J, A_\beta^1, \beta),$$

with \mathcal{P} the map (6.2) in the case $J = J^\lambda$. (Note that there is an analogous bijection $\mathcal{O}(R^\lambda, \beta) \rightarrow \overline{\mathcal{M}}_{1,1}^n(J, A_\beta^n, \beta)$, for $n > 1$, where $A_\beta^n = n \cdot \beta \otimes [S^1]$).

In the particular case of J^λ , we see that all elliptic curves in $C \times S^1$ are Reeb tori, and hence the underlying complex structure on the domain is “rectangular”. That is, they are quotients of the complex plane by a rectangular lattice. This stops being the case when we consider generalized Reeb tori in Section 7.1 for the mapping torus of some strict contactomorphism. Moreover, for more general compatible complex structures we might have nodal degenerations.

Proof of Proposition 6.3. We define $\mathcal{P}(o)$ to be the class represented by the unique up to isomorphism J -holomorphic curve $u : T^2 \rightarrow M$ determined by the conditions:

²It is in fact an equivalence of the corresponding topological action groupoids, but we do not need this explicitly.

- u is charge 1.
- The image of u is the image \mathcal{T} of the map $u_o : T^2 \rightarrow M$, $(s, t) \rightarrow (o(s), t)$, i.e. the image of the Reeb torus of o .
- The degree of the map $u : T^2 \rightarrow \mathcal{T}$ is the multiplicity of o .

We need to show that \mathcal{P} is bijective. Injectivity is automatic. Suppose we have a curve $u \in \overline{\mathcal{M}}_{1,1}^1(J, A, \beta)$, represented by $u : \Sigma \rightarrow M$. By Lemma 6.1 u is a Reeb 2-curve. Then u has no spherical components, as such a component corresponds to a J^λ -holomorphic map $u' : \mathbb{C}\mathbb{P}^1 \rightarrow M$, which by Lemma 6.1 is also a Reeb 2-curve, and this is impossible by second property in the definition.

We first show that u is a finite covering map onto the image of some Reeb torus u_o .

By Lemma 7.7 normalization \tilde{u} is also a Reeb 2-curve. If u is not normal then \tilde{u} is a Reeb 2-curve with domain $\mathbb{C}\mathbb{P}^1$, which is impossible by the argument above. Hence u is normal.

By the charge 1 condition $pr_{S^1} \circ u$ is surjective, where $pr_{S^1} : C \times S^1 \rightarrow S^1$ is the projection. By the Sard theorem we have a regular value $t_0 \in S^1$, so that $u^{-1} \circ pr_{S^1}^{-1}(t_0)$ contains an embedded circle $S_0 \subset \Sigma$. Now $d(pr_{S^1} \circ u)$ is surjective onto $T_{t_0}S^1$ along $T\Sigma|_{S_0}$. And so by first property of u being a Reeb 2-curve, $o = pr_C \circ u|_{S_0}$ has non-vanishing differential $d(o)$. Moreover, again by the first property, o is tangent to $\ker d\lambda$. It follows that o is an unparametrized λ -Reeb orbit.

Also, the image of $d(pr_C \circ u)$ is in $\ker d\lambda$ from which it follows that $\text{image } d(pr_C \circ u) = \text{image } d(o)$. By Sard's theorem and by basic differential topology it follows that the image of u is contained in the image of the Reeb torus u_o , which is an embedded 2-torus \mathcal{T} .

By J^λ -holomorphicity of u , since $\Sigma \simeq T^2$, and by basic complex analysis of holomorphic maps $T^2 \rightarrow T^2$, u is a holomorphic covering map onto \mathcal{T} , of degree $\deg u$.

Let \tilde{o} be $\deg u$ cover of o . Then $\mathcal{P}(\tilde{o})$ is also represented by a degree $\deg u$, charge one holomorphic covering map $u' : T^2 \rightarrow \mathcal{T}$. By basic covering map theory there is a homeomorphism of covering spaces:

$$\begin{array}{ccc} T^2 & \xrightarrow{f} & T^2 \\ \downarrow u & \swarrow u' & \\ \mathcal{T} & & \end{array} .$$

Then f is a biholomorphism, so that u, u' are equivalent. \square

Proposition 6.4. *Let (C, ξ) be a general contact manifold. If λ is a non-degenerate contact 1-form for ξ then all the elements of $\overline{\mathcal{M}}_{1,1}^1(J^\lambda, A, \beta)$ are regular curves. Moreover, if λ is degenerate then for a period c Reeb orbit o , the kernel of the associated real linear Cauchy-Riemann operator for the Reeb torus u_o is naturally identified with the 1-eigenspace of $\phi_{c,*}^\lambda$ - the time c linearized return map $\xi(o(0)) \rightarrow \xi(o(0))$ induced by the R^λ Reeb flow.*

Proof. We already know by Proposition 6.3 that all $u \in \overline{\mathcal{M}}_{1,1}^1(J^\lambda, A, \beta)$ are equivalent to Reeb tori. In particular, such curves have a representation by a J^λ -holomorphic map

$$u : (T^2, j) \rightarrow (Y = C \times S^1, J^\lambda).$$

Since each u is immersed we may naturally get a splitting $u^*T(Y) \simeq N \times T(T^2)$, using the g_J metric, where $N \rightarrow T^2$ denotes the pull-back, of the g_J -normal bundle to image u , and which is identified with the pullback of the distribution ξ_λ on Y , (which we also call the co-vanishing distribution).

The full associated real linear Cauchy-Riemann operator takes the form:

$$(6.5) \quad D_u^J : \Omega^0(N \oplus T(T^2)) \oplus T_j M_{1,1} \rightarrow \Omega^{0,1}(T(T^2), N \oplus T(T^2)).$$

This is an index 2 Fredholm operator (after standard Sobolev completions), whose restriction to $\Omega^0(N \oplus T(T^2))$ preserves the splitting, that is the restricted operator splits as

$$D \oplus D' : \Omega^0(N) \oplus \Omega^0(T(T^2)) \rightarrow \Omega^{0,1}(T(T^2), N) \oplus \Omega^{0,1}(T(T^2), T(T^2)).$$

On the other hand the restricted Fredholm index 2 operator

$$\Omega^0(T(T^2)) \oplus T_j M_{1,1} \rightarrow \Omega^{0,1}(T(T^2)),$$

is surjective by classical Teichmuller theory, see also [29, Lemma 3.3] for a precise argument in this setting. It follows that D'_u will be surjective if the restricted Fredholm index 0 operator

$$D : \Omega^0(N) \rightarrow \Omega^{0,1}(N),$$

has no kernel.

The bundle N is symplectic with symplectic form on the fibers given by restriction of $u^*d\lambda$, and together with J^λ this gives a Hermitian structure (g_λ, j_λ) on N . We have a linear symplectic connection \mathcal{A} on N , which over the slices $S^1 \times \{t\} \subset T^2$ is induced by the pullback by u of the linearized R^λ Reeb flow. Specifically the \mathcal{A} -transport map from the fiber $N_{(s_0,t)}$ to the fiber $N_{(s_1,t)}$ over the path $[s_0, s_1] \times \{t\} \subset T^2$, is given by

$$(u_*|_{N_{(s_1,t)}})^{-1} \circ (\phi_{c(s_1-s_0)}^\lambda)_* \circ u_*|_{N_{(s_0,t)}},$$

where $\phi_{c(s_1-s_0)}^\lambda$ is the time $c \cdot (s_1 - s_0)$ map for the R^λ Reeb flow, where c is the period of the Reeb orbit o_u , and where $u_* : N \rightarrow TY$ denotes the natural map, (it is the universal map in the pull-back diagram.)

The connection \mathcal{A} is defined to be trivial in the θ_2 direction, where trivial means that the parallel transport maps are the *id* maps over θ_2 rays. In particular the curvature $R_{\mathcal{A}}$, understood as a Lie algebra valued 2-form, of this connection vanishes. The connection \mathcal{A} determines a real linear CR operator $D_{\mathcal{A}}$ on N in the standard way, take the complex anti-linear part of the vertical differential of a section. Explicitly,

$$D_{\mathcal{A}} : \Omega^0(N) \rightarrow \Omega^{0,1}(N),$$

is defined by

$$D_{\mathcal{A}}(\mu)(p) = j_\lambda \circ \pi^{vert}(\mu(p)) \circ d\mu(p) - \pi^{vert}(\mu(p)) \circ d\mu(p) \circ j,$$

where

$$\pi^{vert}(\mu(p)) : T_{\mu(p)}N \rightarrow T_{\mu(p)}^{vert}N \simeq N$$

is the \mathcal{A} -projection, and where $T_{\mu(p)}^{vert}N$ is the kernel of the projection $T_{\mu(p)}N \rightarrow T_p\Sigma$. It is elementary to verify that the operator $D_{\mathcal{A}}$ is Fredholm 0 with the kernel isomorphic to the kernel of D . See also [20, Section 10.1] for a computation of this kind in much greater generality.

We have a differential 2-form Ω on the total space of N defined as follows. On the fibers $T^{vert}N$, $\Omega = u_*\omega$, for $\omega = d_\alpha\lambda$, and for $T^{vert}N \subset TN$ denoting the vertical tangent space, or subspace of vectors v with $\pi_*v = 0$, for $\pi : N \rightarrow T^2$ the projection. While on the \mathcal{A} -horizontal distribution Ω is defined to vanish. The 2-form Ω is closed, which we may check explicitly by using that $R_{\mathcal{A}}$ vanishes to obtain local symplectic trivialisations of N in which \mathcal{A} is trivial. Clearly Ω must vanish on the 0-section since it is a \mathcal{A} -flat section. But any section is homotopic to the 0-section and so in particular if $\mu \in \ker D$ then Ω vanishes on μ .

Since $\mu \in \ker D$, and so its vertical differential is complex linear, it follows that the vertical differential vanishes. To see this note that $\Omega(v, J^\lambda v) > 0$, for $0 \neq v \in T^{vert}N$ and so if the vertical differential did not vanish we would have $\int_\mu \Omega > 0$. So μ is \mathcal{A} -flat, in particular the restriction of μ over all slices $S^1 \times \{t\}$ is identified with a period c orbit of the linearized at o R^λ Reeb flow, and which does not depend on t as \mathcal{A} is trivial in the t variable. So the kernel of D is identified with the vector space of period c orbits of the linearized at o R^λ Reeb flow, as needed. \square

Proposition 6.6. *Let λ be a contact form on a $(2n + 1)$ -fold C , and o a non-degenerate, period c , λ -Reeb orbit, then the orientation of $[u_o]$ induced by the determinant line bundle orientation of $\overline{\mathcal{M}}_{1,1}^1(J^\lambda, A)$, is $(-1)^{CZ(o)-n}$, which is*

$$\text{sign Det}(\text{Id}|_{\xi(o(0))} - \phi_{c,*}^\lambda|_{\xi(o(0))}).$$

Proof of Proposition 6.6. Abbreviate u_o by u . Let $N \rightarrow T^2$ be the vector bundle associated to u as in the proof of Proposition 6.4. Fix a trivialization ϕ of N induced by any trivialization of the contact distribution ξ along o in the obvious sense: N is the pullback of ξ along the composition

$$T^2 \rightarrow S^1 \xrightarrow{o} C.$$

Let the symplectic connection \mathcal{A} on N be defined as before. Then the pullback connection $\mathcal{A}' := \phi^* \mathcal{A}$ on $T^2 \times \mathbb{R}^{2n}$ is a connection whose parallel transport paths $p_t : [0, 1] \rightarrow \text{Symp}(\mathbb{R}^{2n})$, along the closed loops $S^1 \times \{t\}$, are paths starting at 1, and are t independent. And so the parallel transport path of \mathcal{A}' along $\{s\} \times S^1$ is constant, that is \mathcal{A}' is trivial in the t variable. We shall call such a connection \mathcal{A}' on $T^2 \times \mathbb{R}^{2n}$ induced by p .

By non-degeneracy assumption on o , the map $p(1)$ has no 1-eigenvalues. Let $p'' : [0, 1] \rightarrow \text{Symp}(\mathbb{R}^{2n})$ be a path from $p(1)$ to a unitary map $p''(1)$, with $p''(1)$ having no 1-eigenvalues, and s.t. p'' has only simple crossings with the Maslov cycle. Let p' be the concatenation of p and p'' . We then get

$$CZ(p') - \frac{1}{2} \text{sign } \Gamma(p', 0) \equiv CZ(p') - n \equiv 0 \pmod{2},$$

since p' is homotopic relative end points to a unitary geodesic path h starting at id , having regular crossings, and since the number of negative, positive eigenvalues is even at each regular crossing of h by unitarity. Here $\text{sign } \Gamma(p', 0)$ is the index of the crossing form of the path p' at time 0, in the notation of [23]. Consequently,

$$(6.7) \quad CZ(p'') \equiv CZ(p) - n \pmod{2},$$

by additivity of the Conley-Zehnder index.

Let us then define a free homotopy $\{p_t\}$ of p to p' , p_t is the concatenation of p with $p''|_{[0,t]}$, reparametrized to have domain $[0, 1]$ at each moment t . This determines a homotopy $\{\mathcal{A}'_t\}$ of connections induced by $\{p_t\}$. By the proof of Proposition 6.4, the CR operator D_t determined by each \mathcal{A}'_t is surjective except at some finite collection of times $t_i \in (0, 1)$, $i \in N$ determined by the crossing times of p'' with the Maslov cycle, and the dimension of the kernel of D_{t_i} is the 1-eigenspace of $p''(t_i)$, which is 1 by the assumption that the crossings of p'' are simple.

The operator D_1 is not complex linear. To fix this we concatenate the homotopy $\{D_t\}$ with the homotopy $\{\tilde{D}_t\}$ defined as follows. Let $\{\tilde{\mathcal{A}}_t\}$ be a homotopy of \mathcal{A}'_1 to a unitary connection $\tilde{\mathcal{A}}_1$, where the homotopy $\{\tilde{\mathcal{A}}_t\}$ is through connections induced by paths $\{\tilde{p}_t\}$, giving a path homotopy of $p' = \tilde{p}_0$ to h . Then $\{\tilde{D}_t\}$ is defined to be induced by $\{\tilde{\mathcal{A}}_t\}$.

Let us denote by $\{D'_t\}$ the concatenation of $\{D_t\}$ with $\{\tilde{D}_t\}$. By construction, in the second half of the homotopy $\{D'_t\}$, D'_t is surjective. And D'_1 is induced by a unitary connection, since it is induced by unitary path \tilde{p}_1 . Consequently, D'_1 is complex linear. By the above construction, for the homotopy $\{D'_t\}$, D'_t is surjective except for N times in $(0, 1)$, where the kernel has dimension one. In particular the sign of $[u]$ by the definition via the determinant line bundle is exactly

$$-1^N = -1^{CZ(p)-n},$$

by (6.7), which was what to be proved. \square

Theorem 6.8.

$$GW_{1,1}^1(N, J^\lambda, A_\beta, \beta)([\overline{\mathcal{M}}_{1,1}] \otimes [C \times S^1]) = i(\mathcal{P}^{-1}(N), R^\lambda, \beta),$$

where $N \subset \overline{\mathcal{M}}_{1,1}^1(J^\lambda, A_\beta, \beta)$ is an open compact set (where \mathcal{P} is as in Proposition 6.3), $i(\mathcal{P}^{-1}(N), R^\lambda, \beta)$ is the Fuller index as described in the Appendix below, and where the left-hand side of the equation is the functional as in (5.2).

Proof. Suppose that $N \subset \overline{\mathcal{M}}_{1,1}^1(J^\lambda, A_\beta, \beta)$ is open-compact and consists of isolated regular Reeb tori $\{u_i\}$, corresponding to orbits $\{o_i\}$. Denote by $\text{mult}(o_i)$ the multiplicity of the orbits as in Appendix A. Then we have:

$$GW_{1,1}^1(N, J^\lambda, A_\beta, \beta)([\overline{\mathcal{M}}_{1,1}] \otimes [C \times S^1]) = \sum_i \frac{(-1)^{CZ(o_i)-n}}{\text{mult}(o_i)},$$

where n half the dimension of M , the numerator is as in (A.2), and $\text{mult}(o_i)$ is the order of the corresponding isotropy group, see Appendix B.

The expression on the right is exactly the Fuller index $i(\mathcal{P}^{-1}(N), R^\lambda, \beta)$. Thus, the theorem follows for N as above. However, in general if N is open and compact then perturbing slightly we obtain a smooth family $\{R^{\lambda_t}\}$, $\lambda_0 = \lambda$, s.t. λ_1 is non-degenerate, that is has non-degenerate orbits. And such that there is an open-compact subset \tilde{N} of $\overline{\mathcal{M}}_{1,1}^1(\{J^{\lambda_t}\}, A_\beta, \beta)$ with $(\tilde{N} \cap \overline{\mathcal{M}}_{1,1}^1(J^\lambda, A_\beta, \beta)) = N$, see Lemma 5.9. Then by Lemma 5.3 if

$$N_1 = (\tilde{N} \cap \overline{\mathcal{M}}_{1,1}^1(J^{\lambda_1}, A_\beta, \beta))$$

we get

$$GW_{1,1}^1(N, J^\lambda, A_\beta, \beta)([\overline{\mathcal{M}}_{1,1}] \otimes [C \times S^1]) = GW_{1,1}^1(N_1, J^{\lambda_1}, A_\beta, \beta)([\overline{\mathcal{M}}_{1,1}] \otimes [C \times S^1]).$$

By the previous discussion

$$GW_{1,1}^1(N_1, J^{\lambda_1}, A_\beta, \beta)([\overline{\mathcal{M}}_{1,1}] \otimes [C \times S^1]) = i(N_1, R^{\lambda_1}, \beta),$$

but by the invariance of Fuller index (see Appendix A),

$$i(N_1, R^{\lambda_1}, \beta) = i(N, R^\lambda, \beta).$$

□

What about higher genus invariants of $C \times S^1$? Following the proof of Proposition 6.3, it is not hard to see that all J^λ -holomorphic curves must be branched covers of Reeb tori. If one can show that these branched covers are regular when the underlying tori are regular, the calculation of invariants would be fairly automatic from this data. See [30], [28] where these kinds of regularity calculation are made.

7. PROOFS OF MAIN THEOREMS

To set notation and terminology we review the basic definition of a nodal curve.

Definition 7.1. A nodal Riemann surface (without boundary) is a pair $\Sigma = (\tilde{\Sigma}, \mathcal{N})$ where $\tilde{\Sigma}$ is a Riemann surface, and \mathcal{N} a set of pairs of points of $\tilde{\Sigma} : \mathcal{N} = \{(z_0^0, z_0^1), \dots, (z_n^0, z_n^1)\}$, $n_i^j \neq n_k^l$ for $i \neq k$ and all j, l . By slight abuse, we may also denote by Σ the quotient space $\tilde{\Sigma}/\sim$, where the equivalence relation is generated by $n_i^0 \sim n_i^1$. Let $q_\Sigma : \tilde{\Sigma} \rightarrow (\tilde{\Sigma}/\sim)$ denote the quotient map. The elements $q_\Sigma(\{z_i^0, z_i^1\}) \in \tilde{\Sigma}/\sim$, are called **nodes**. Let M be a smooth manifold. By a map $u : \Sigma \rightarrow M$ of a nodal Riemann surface Σ , we mean a set map $u : (\tilde{\Sigma}/\sim) \rightarrow M$. u is called smooth or immersion or J -holomorphic (when M is almost complex) if the map $\tilde{u} = u \circ q_\Sigma$ is smooth or respectively immersion or respectively J -holomorphic. We call \tilde{u} **normalization of u** . u is called an embedding if u is a topological embedding and its normalization is an immersion. The cohomology groups of Σ are defined as $H^\bullet(\Sigma) := H^\bullet(\tilde{\Sigma}/\sim)$, likewise with homology. The genus of Σ is the topological genus of $\tilde{\Sigma}/\sim$.

We shall say that $(\tilde{\Sigma}, \mathcal{N})$ is normal if $\mathcal{N} = \emptyset$. Similarly, $u : \Sigma \rightarrow M$, $\Sigma = (\tilde{\Sigma}, \mathcal{N})$ is called **normal** if $\mathcal{N} = \emptyset$. The normalization of u is the map of the nodal Riemann surface $\tilde{u} : \tilde{\Sigma} \rightarrow M$, $\tilde{\Sigma} = (\tilde{\Sigma}, \emptyset)$. Note that if u is a Reeb 2-curve, its normalization \tilde{u} may not be a Reeb 2-curve (the second condition may fail).

7.1. Mapping tori and Reeb 2-curves. Let (C, λ) be a contact manifold and ϕ a strict contactomorphism and let $M = (M_{\phi,1}, \lambda_{\phi}, \alpha)$ denote the mapping torus of ϕ , as also appearing in Theorem 2.8. More specifically, $M = C \times \mathbb{R} / \sim$, where the equivalence \sim is generated by $(x, \theta) \sim (\phi(x), \theta + 1)$, for more details on the corresponding lcs structure see for instance [3]. Then $(M, \lambda_{\phi}, \alpha)$ is an integral first kind lcs manifold.

In this case $\mathcal{V}_{\lambda} = \mathcal{D}$, and in mapping torus coordinates at a point (x, θ) , it is spanned by $X_{\lambda} = (0, \frac{\partial}{\partial \theta})$, $X_{\alpha} = (R^{\lambda_{\theta}}, 0)$ for $R^{\lambda_{\theta}}$ the λ_{θ} -Reeb vector field, where $\lambda_{\theta} = \lambda_{C_{\theta}}$ the fiber over θ of the projection $M \rightarrow S^1$. Analogously to the Example 5 there is an S^1 -invariant almost complex structure on M , which we call $J^{\lambda_{\phi}}$.

We now show that all Reeb 2-curves in M must be of a certain type. Let $o : S^1 \rightarrow C$ be a λ -Reeb and suppose that $\text{image } \phi^n(o) = \text{image } o$, for some $n > 0$, so that

$$\forall t \in [0, 1] : \phi^n(o)(t) = o(t + \theta_0)$$

for some uniquely determined $\theta_0 \in [0, 1)$. Let $\tilde{o} : S^1 \times [0, n] \rightarrow C \times \mathbb{R}$ be the map

$$\tilde{o}(t, \tau) = (o(t + \theta_0 \cdot \frac{\tau}{n}), \tau).$$

Then \tilde{o} is well defined on the quotient $T^2 \simeq S^1 \times ([0, n]/0 \sim n)$, and we denote the quotient map by u_o^n , called the *charge n generalized Reeb torus of o* . If the class $[o] \in \pi_1(C) = \beta$ we denote by A_{β}^n the class of u_o^n in $H_2(M, \mathbb{Z})$. The class A_{β}^n can be defined more generally whenever $\phi_*^n(\beta) = \beta$, it is the class of a torus map $T^2 \rightarrow M$ defined analogously to the map u_o^n , but no longer having the Reeb 2-curve property. We may abbreviate A_{β}^1 by A_{β} .

By construction, u_o^n is a charge n Reeb 2-curve and its image is an embedded $J^{\lambda_{\phi}}$ -holomorphic torus \mathcal{T} . Moreover, u_o^n is $J^{\lambda_{\phi}}$ -holomorphic with respect to a uniquely determined complex structure on T^2 , similarly to the case of Reeb tori of Section 4.2. However, unlike the case of Reeb tori, this complex structure is not “rectangular” unless $\phi^n \circ o = o$.

Proposition 7.2. *Let $M = (M_{\phi}, \lambda_{\phi}, \alpha)$ be the mapping torus of a strict contactomorphism ϕ as above. Then:*

- (1) *Every charge n Reeb 2-curve u in M has a factorization:*

$$(7.3) \quad u = u_o^n \circ \rho,$$

for $\rho : \Sigma \rightarrow T^2$ some degree one map, and for some orbit string o uniquely determined by u .

- (2) *Every element $u \in \overline{\mathcal{M}}_{1,1}^n(J^{\lambda_{\phi}}, A_{\beta}^n, \beta)$ is represented (as an equivalence class in this moduli space) by u_o^n , where the latter is as above, for some o uniquely determined.*
- (3) *The Fredholm index of the corresponding real linear CR operator is 2, so that the expected dimension of $\overline{\mathcal{M}}_{1,1}^n(J^{\lambda_{\phi}}, A_{\beta}^n, \beta)$ is 0.*
- (4) *Let J be a (λ_{ϕ}, α) -admissible almost complex structure on M . There is a natural proper topological embedding:*

$$\text{emb} : \overline{\mathcal{M}}_{1,1}^n(J, A_{\beta}^n, \beta) \rightarrow \mathcal{O}(R^{\lambda}, \beta),$$

defined by $u \mapsto o$, where o is uniquely determined by the condition (7.3).

Proof. The proof of part one is completely analogous to the proof of Proposition 6.3. To prove the second part, first note that by the first part u has image $\mathcal{T} = \text{image } u_o^n$ for some n, o , and \mathcal{T} is an embedded J^{λ_ϕ} -holomorphic torus. By basic theory of mappings of complex tori u must be a covering map $\Sigma \rightarrow \mathcal{T}$. Since we know the charge n , as in final part of the proof of Proposition 6.3, we may conclude that $u \simeq u_o^n$ for some uniquely determined o , where \simeq is an isomorphism.

We prove Part 3. Note that $c_1(A_\beta^n) = 0$, as by construction the complex tangent bundle along u_o^n admits a flat connection, induced by the natural \mathcal{G} -connection on $M_\phi \rightarrow S^1$, for \mathcal{G} the group of strict contactomorphisms of (C, λ) , cf. Proof of Proposition 6.4. The needed fact then follows by the index/Riemann-Roch theorem.

The last part of the proposition readily follows from the first part. \square

Proof of Theorem 1.10. Let $u : \Sigma \rightarrow (M = M_{\tilde{\phi}, 1})$ be a Reeb 2-curve in the mapping torus as in the statement. By part one of Proposition 7.2, there must be a generalized charge n Reeb torus in M . By definitions this means that $\tilde{\phi}$ has a charge n fixed Reeb string, so that ϕ has a charge n fixed geodesic string, which is impossible by assumptions. \square

Proposition 7.4. *Let (C, λ) be a contact manifold with λ satisfying one of the following conditions:*

- (1) *There is a non-degenerate λ -Reeb orbit.*
- (2) *$i(N, R^\lambda, \beta) \neq 0$ for some open compact $N \subset \mathcal{O}(R^\lambda, \beta)$, and some β .*

Then:

- (1) *Let (λ, α) be the lcs-ification of (C, λ) . There exists an $\epsilon > 0$ s.t. for any tamed exact lcs structure (λ', α', J) on $M = C \times S^1$, with $(d_{\alpha'}\lambda', J)$ ϵ -close to $(d_\alpha\lambda, J^\lambda)$ (as in Definition 5.6), there exists an elliptic, J -holomorphic α -charge 1 curve u in M .*
- (2) *In addition, if (M, λ', α') is first kind and has dimension 4 then u may be assumed to be normal and embedded.*

Proof. If we have a closed non-degenerate λ -Reeb orbit o then we also have an open compact subset $N = \{o\} \subset S_\lambda$. Thus suppose that the condition 2 holds.

Set

$$(\tilde{N} := \mathcal{P}(N)) \subset \overline{\mathcal{M}}_{1,1}^1(J^\lambda, A_\beta, \beta),$$

which is an open compact set. By Theorem 6.8, and by the assumption that $i(N, R_\lambda, \beta) \neq 0$

$$GW_{1,1}^1(N, J^\lambda, A_\beta, \beta) \neq 0.$$

The first part of the conclusion then follows by Proposition 5.11.

We now verify the second part. Suppose that M has dimension 4. Let U be an ϵ -neighborhood of $(\lambda, \alpha, J^\lambda)$, for ϵ as given in the first part, and let $(\lambda', \alpha', J) \in U$. Suppose that $u \in \overline{\mathcal{M}}_{1,1}^1(J, \beta)$. Let \underline{u} be a simple J -holomorphic curve covered by u , (see for instance [18, Section 2.5]).

For convenience, we now recall the adjunction inequality.

Theorem 7.5 (McDuff-Micallef-White [19], [14]). *Let (M, J) be an almost complex 4-manifold and let $A \in H_2(M)$ be a homology class that is represented by a simple J -holomorphic curve $u : \Sigma \rightarrow M$. Let $\delta(u)$ denote the number of self-intersections of u , then*

$$2\delta(u) - \chi(\Sigma) \leq A \cdot A - c_1(A),$$

with equality if and only if u is an immersion with only transverse self-intersections.

In our case $A = A_\beta$ so that $c_1(A) = 0$ and $A \cdot A = 0$. If u is not normal its normalization is of the form $\tilde{u} : \mathbb{C}\mathbb{P}^1 \rightarrow M$ with at least one self intersection and with $0 = [\tilde{u}] \in H_2(M)$, but this contradicts positivity of intersections. So u and hence \underline{u} are normal. Moreover, the domain Σ' of \underline{u} satisfies: $\chi(\Sigma') = \chi(T^2) = 0$, so that $\delta(\underline{u}) = 0$, and the above inequality is an equality. In particular \underline{u} is an embedding, which of course implies our claim. □

Proof of Theorem 3.3. Let

$$U \ni (\omega_0 := d_\alpha \lambda, J_0 := J^\lambda)$$

be a set of pairs (ω, J) satisfying the following:

- ω is a first kind lcs structure.
- For each $(\omega, J) \in U$, J is ω -compatible and admissible.
- Let ϵ be chosen as in the first part of Proposition 7.4. Then each $(\omega, J) \in U$ is ϵ -close to (ω_0, J_0) , (as in Definition 5.6).

To prove the theorem we need to construct a map $E : V \rightarrow \mathcal{J}(M)$, where V is some neighborhood of ω_0 in the space $(\mathcal{F}(M), d_\infty)$ (see Definition 3.1) and where

$$\forall \omega \in V : (\omega, E(\omega)) \in U.$$

As then Proposition 7.4 tells us that for each $\omega \in V$, there is a class A , $E(\omega)$ -holomorphic, elliptic curve u in M . Using Lemma 6.1 we would then conclude that there is an elliptic Reeb 2-curve u in (M, ω) . If M has dimension 4 then in addition u may be assumed to be normal and embedded. If ω is integral, by Proposition 7.4, u may be assumed to be charge 1. And so we will be done.

Define a metric ρ_0 measuring the distance between subspaces W_1, W_2 , of same dimension, of an inner product space (T, g) as follows.

$$\rho_0(W_1, W_2) := |P_{W_1} - P_{W_2}|,$$

for $|\cdot|$ the g -operator norm, and P_{W_i} g -projection operators onto W_i .

Let $\delta > 0$ be given. Suppose that $\omega = d^{\alpha'} \lambda'$ is a first kind lcs structure δ -close to ω_0 for the metric d_∞ . Then $\mathcal{V}_{\lambda'}, \xi_{\lambda'}$ are smooth distributions by the assumption that (α', λ') is a lcs structure of the first kind and $TM = \mathcal{V}_{\lambda'} \oplus \xi_{\lambda'}$. Moreover,

$$\rho_\infty(\mathcal{V}_{\lambda'}, \mathcal{V}_\lambda) < \epsilon_\delta$$

and

$$\rho_\infty(\xi_{\lambda'}, \xi_\lambda) < \epsilon_\delta$$

where $\epsilon_\delta \rightarrow 0$ as $\delta \rightarrow 0$, and where ρ_∞ is the C^∞ analogue of the metric ρ_0 , for the family of subspaces of the family of inner product spaces $(T_p M, g)$.

Then choosing δ to be suitably small, for each $p \in M$ we have an isomorphism

$$\phi(p) : T_p M \rightarrow T_p M,$$

$\phi_p := P_1 \oplus P_2$, for $P_1 : \mathcal{V}_{\lambda_0}(p) \rightarrow \mathcal{V}_{\lambda'}(p)$, $P_2 : \xi_{\lambda_0}(p) \rightarrow \xi_{\lambda'}(p)$ the g -projection operators. Define $E(\omega)(p) := \phi(p)_* J_0$. Then clearly, if δ was chosen to be sufficiently small, if we take V to be the δ -ball in $(\mathcal{F}(M), d_\infty)$ centered at ω_0 , then it has the needed property. □

Definition 7.6. Let α be a scale integral closed 1-form on a closed smooth manifold M . Let $0 \neq c \in \mathbb{R}$ be such that $c\alpha$ is integral. A **classifying map** $p : M \rightarrow S^1$ of α is a smooth map s.t. $c\alpha = p^* d\theta$. A map p with these properties is of course not unique.

Lemma 7.7. *Let $u : \Sigma \rightarrow M$ be a Reeb 2-curve in a closed, scale integral, first kind lcs manifold (M, λ, α) , then its normalization $\tilde{u} : \tilde{\Sigma} \rightarrow M$ is a Reeb 2-curve.*

Proof. By Lemma 3.6 we have a surjective classifying map $p : M \rightarrow S^1$ of α . Note that the fibers M_t of p , for all $t \in S^1$, are contact with contact form $\lambda_t = \lambda|_{C_t}$, as $0 \neq \omega^n = \alpha \wedge \lambda \wedge d\lambda^{n-1}$ and $c \cdot \alpha = 0$ on M_t , where c is as in the definition of p .

Let $\tilde{u} : \tilde{\Sigma} \rightarrow M$ be the normalization of u . Suppose it is not a Reeb 2-curve, which in this case, by definitions, just means that $0 = [\tilde{u}^*\alpha] \in H^1(\tilde{\Sigma}, \mathbb{R})$. Since $0 \neq [u^*\alpha] \in H^1(\Sigma, \mathbb{R})$, some node z_0 of Σ lies on closed loop $o : S^1 \rightarrow \Sigma$ with $\langle [o], [u^*\alpha] \rangle \neq 0$.

Let $q_\Sigma : \tilde{\Sigma} \rightarrow \Sigma$ be the quotient map as previously appearing. In this case, we may find a smooth embedding $\eta : D^2 \rightarrow \tilde{\Sigma}$, s.t. $q_\Sigma \circ \eta|_{\partial D^2}$ is a component of a regular fiber C_t , of the classifying map $p' : \Sigma \rightarrow S^1$ of $u^*\alpha$. See Figure 1, $\eta(D^2)$ is a certain disk in $\tilde{\Sigma}$, whose interior contains an element of $\phi^{-1}(z_0)$.

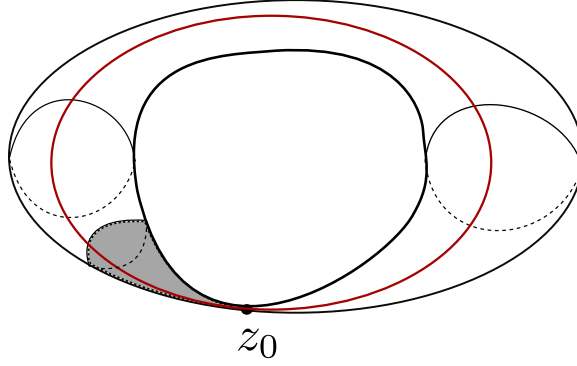


FIGURE 1. The figure for Σ . The gray shaded area is the image $q_\Sigma \circ \eta(D^2)$. The red shaded curve is the image of the closed loop o as above.

Then analogously to the proof of Proposition 6.3 $\tilde{u} \circ \eta|_{\partial D^2}$ is a (unparametrized) λ_t -Reeb orbit in M_t . (The classifying maps can be arranged, such that $u(C_t) \subset M_t$.) And in particular $\int_{\partial D^2} \tilde{u}^*\lambda \neq 0$. Now u (not \tilde{u}) is a Reeb 2-curve, and the first condition of this implies that $\int_D d\tilde{u}^*\lambda = 0$, since $\ker d\lambda$ on M is spanned by X_λ, X_α . So we have a contradiction to Stokes theorem. Thus, \tilde{u} must be a Reeb 2-curve. □

Proof of Theorem 1.7. Let (C, λ) and ϕ be as in the hypothesis. Let $\{\phi_t\}$, $t \in [0, 1]$ be a smooth family of strict contactomorphisms $\phi_0 = id$, $\phi_1 = \phi$. This gives a smooth fibration $\tilde{M} \rightarrow [0, 1]$, with fiber over $t \in [0, 1]$: $M_{\phi_t, 1}$, which is moreover endowed with the first kind lcs structure $(\lambda_{\phi_t}, \alpha)$, where this is the “mapping torus structure” as just discussed. Let $tr : \tilde{M} \rightarrow (C \times S^1) \times [0, 1]$ be a smooth trivialization, restricting to the identity $C \times S^1 \rightarrow C \times S^1$ over 0. Pushing forward by the bundle map tr , the above mentioned family of lcs structures, we get a smooth family $\{(\lambda_t, \alpha)\}$, $t \in [0, 1]$, of first kind integral lcs structures on $C \times S^1$, with $(\lambda_0, \alpha) = (\lambda, \alpha)$ the standard lcs-fication of λ .

Fix a family $\{J^{\lambda_t}\}$ of almost complex structures on $C \times S^1$ with each J^{λ_t} admissible with respect (λ_t, α) . Let $N \subset \mathcal{O}(R^\lambda, \beta)$ be an open compact set satisfying $i(N, R^\lambda, \beta) \neq 0$. The embedding emb from part 4 of Proposition 7.2 induces a proper embedding

$$\widetilde{emb} : \widetilde{\mathcal{M}} = \overline{\mathcal{M}}_{1,1}^1(\{J^{\lambda_t}\}, A_\beta, \beta) \rightarrow \mathcal{O}(R^\lambda, \beta) \times [0, 1],$$

defined by $\widetilde{emb}(u, t) = (emb(u), t)$.

Set $N_0 = \mathcal{P}(N) \subset \overline{\mathcal{M}}_{1,1}^1(J^{\lambda_0}, A_\beta, \beta) \subset \widetilde{\mathcal{M}}$. So that by construction $\widetilde{emb}(N_0) = N \times \{0\}$. Set

$$\widetilde{N} = \widetilde{emb}^{-1}(\widetilde{emb}(\widetilde{\mathcal{M}}) \cap (N \times [0, 1])),$$

then this is an open and compact subset of $\widetilde{\mathcal{M}}$. And by construction $\widetilde{N} \cap \overline{\mathcal{M}}_{1,1}^1(J^{\lambda_0}, A_\beta, \beta) = N_0$. Set $N_1 = \widetilde{N} \cap \overline{\mathcal{M}}_{1,1}^1(J^{\lambda_1}, A_\beta, \beta)$.

It follows by Theorem 6.8 that

$$GW_{1,1}^1(N_0, J^{\lambda_0}, A_\beta, \beta)([\overline{\mathcal{M}}_{1,1}] \otimes [C \times S^1]) = i(N, R^\lambda, \beta) \neq 0.$$

Then applying Lemma 5.3 we get that

$$GW_{1,1}^1(N_1, J^{\lambda_1}, A_\beta, \beta)([\overline{\mathcal{M}}_{1,1}] \otimes [C \times S^1]) \neq 0.$$

By part two of Proposition 7.2 there is a charge 1 generalized Reeb torus u_o in M . In particular, $\text{image}(\phi(o)) = \text{image}(o)$. Also by construction, $o \in N$ and so we are done. \square

We say that the action spectrum is **discrete** if the image of the period map $A : S(R^\lambda, \beta) \rightarrow \mathbb{R}$, $o \mapsto \int_{S^0} o^* \lambda$ is discrete.

Definition 7.8. *Assuming that the class β is non-torsion³, we say that (C, λ) is **definite type** for class β if the action spectrum of λ is discrete and there is a renormalization $\{X^a\}_{a \in \mathbb{R}_{>0}}$ of \mathbb{R}^λ , by smooth vector fields (not necessarily Reeb) in the sense of [25, Definition 2.6], s.t. all but finitely many class β orbits of each X^a have even Conley-Zehnder index (positive definite type) or or all but finitely many orbits of X have odd Conley-Zehnder index (negative definite type).*

A typical example of definite type is the standard contact form λ_{st} on S^{2k+1} , as shown in [25]. Another example, is the unit cotangent bundle of a non-positive sectional curvature Riemannian manifold.

Theorem 7.9. *Let (C, λ) be a contact manifold of definite type for class β orbits, then every strict contactomorphism ϕ of (C, λ) , homotopic to the id via strict contactomorphisms, has a fixed Reeb string in class β . Furthermore, the same holds for every λ' sufficiently C^1 nearby to λ . In particular, for any contact form λ on S^{2k+1} , sufficiently C^1 nearby to λ_{st} , any strict contactomorphism ϕ of (C, λ) homotopic to the id via strict contactomorphisms has a fixed Reeb string.*

Proof of Theorem 7.9. For the full definition of renormalization we refer the reader to [25, Definition 2.5] (called there perturbation system). WLOG assume that R^λ is positive definite type. In particular, we may find a perturbation X^a of R^λ together with a homotopy X_t , $t \in [0, 1]$, s.t.:

- (1) $X_0 = X^a$, $X_1 = R^\lambda$.
- (2) $\mathcal{O}(X^a, a, \beta) = \{o \in \mathcal{O}(X^a, \beta) \mid A(o) \leq a\}$ is discrete, where A is the period map as in the Introduction.
- (3) Each $o \in \mathcal{O}(X^a, a, \beta)$ is contained in a non-branching open compact subset $K_o \subset \mathcal{O}(\{X_t^a\}, \beta)$. Where the latter means that:
 - (a) $K_o \cap \mathcal{O}(X_1, \beta)$ is connected.
 - (b) For $o, o' \in \mathcal{O}(X^a, a, \beta)$ $K_o = K_{o'}$ or $K_o \cap K_{o'} = \emptyset$.
 - (c) $\mathcal{O}(X_1, \beta) = \cup_o (K_o \cap \mathcal{O}(X_1, \beta))$.
 - (d) $(K_o \cap \mathcal{O}(X^a, \beta)) \subset \mathcal{O}(X^a, a, \beta)$.

³In the torsion case the infinite type condition is more complicated see [25].

$$(4) \sum_{o \in \mathcal{O}(X^a, a, \beta)} i(o) > 0.$$

Set $N := \mathcal{O}(X^a, a, \beta)$, then by the condition 3 and by the non-branching property,

$$N = \sqcup_{i \in \{1, \dots, n\}} \mathcal{O}(X^a, \beta) \cap K_{o_i},$$

disjoint union for some $o_1, \dots, o_n \in \mathcal{O}(X^a, a, \beta)$. Set $\tilde{N} := \cup_{i \in \{1, \dots, n\}} K_{o_i}$. Then $\tilde{N} \cap \mathcal{O}(X^a, \beta) = N$. Set $N_1 := \tilde{N} \cap \mathcal{O}(R^\lambda, \beta)$, then this is an open compact subset of $\mathcal{O}(X_1, \beta)$.

Finally, using invariance of the Fuller index we get that $i(N_1, R^\lambda, \beta) \neq 0$. Then the result follows by Theorem 1.7. \square

Proof of Theorem 1.6. Suppose that λ is Morse-Bott and we have an open compact component $N \subset \mathcal{O}(R^\lambda, \beta)$, with $\chi(N) \neq 0$ so that $i(N, R^\lambda, \beta) \neq 0$, [25, Section 2.1.1]. If λ' is sufficiently C^1 nearby to λ then we may find open compact $N' \in \mathcal{O}(R^{\lambda'})$ s.t.

$$i(N', R^{\lambda'}, \beta) = i(N, R^\lambda, \beta) \neq 0.$$

See [25, Lemma 1.6]. Then the result follows by Theorem 1.7. \square

Proof of Theorem 3.4. Suppose that $u : \Sigma \rightarrow M$ is an immersed Reeb 2-curve, we then show that M also has a Reeb 1-curve. Let $\tilde{u} : \tilde{\Sigma} \rightarrow M$ be the normalization of u , so that \tilde{u} is an immersion. We have a pair of transverse 1-distributions $D_1 = \tilde{u}^* \mathbb{R} \langle X_\alpha \rangle$, $D_2 = \tilde{u}^* \mathbb{R} \langle X_\lambda \rangle$ on $\tilde{\Sigma}$. We may then find an embedded path $\gamma : [0, 1] \rightarrow \tilde{\Sigma}$, tangent to D_1 s.t. $\lambda(\gamma'(t)) > 0$, $\forall t \in [0, 1]$, and s.t. $\gamma(0)$ and $\gamma(1)$ are on a leaf of D_2 . It is then simple to obtain from this a Reeb 1-curve o , by joining the end points of γ by an embedded path tangent to D_2 , and perturbing, see Figure 2. This proves the first part of the theorem.

To prove the second part, suppose that $u : \Sigma \rightarrow M$ is an immersed elliptic Reeb 2-curve. Suppose that u is not normal. Let $\tilde{u} : \tilde{\Sigma} \rightarrow M$ be its normalization. Then $\tilde{\Sigma}$ has a genus 0 component \mathcal{S} , such that $\tilde{u} : \mathcal{S} \simeq \mathbb{C}\mathbb{P}^1 \rightarrow M$ is immersed. The distribution $D_1 = \tilde{u}^* \mathbb{R} \langle X_\alpha \rangle$, as appearing above, is then a $\tilde{u}^* \lambda$ -oriented 1-dimensional distribution on $\mathbb{C}\mathbb{P}^1$ which is impossible. \square

A. FULLER INDEX

Let X be a complete vector field without zeros on a smooth manifold M . Set

$$(A.1) \quad S(X, \beta) = \{o \in L_\beta M \mid \exists p \in (0, \infty), o : \mathbb{R}/\mathbb{Z} \rightarrow M \text{ is a periodic orbit of } pX\},$$

where $L_\beta M$ denotes the free homotopy class β component of the free loop space

$$LM = \{o : S^1 \rightarrow M \mid o \text{ is smooth}\}.$$

And where recall that $S^1 = \mathbb{R}/\mathbb{Z}$. The above p is uniquely determined and we denote it by $p(o)$ called the period of o .

There is a natural S^1 reparametrization action on $S(X, \beta)$: $t \cdot o$ is the loop $t \cdot o(\tau) = o(t + \tau)$. The elements of $\mathcal{O}(X, \beta) := S(X, \beta)/S^1$ will be called *orbit strings*. Slightly abusing notation we just write o for the equivalence class of o .

The multiplicity $m(o)$ of an orbit string is the ratio $p(o)/l$ for $l > 0$ the period of a simple orbit covered by o .

We want a kind of fixed point index of an open compact subset $N \subset \mathcal{O}(X, \beta)$, which counts orbit strings o with certain weights. Assume for simplicity that $N \subset \mathcal{O}(X)$ is finite. (Otherwise, for a general open compact $N \subset \mathcal{O}(X, \beta)$, we need to perturb.) Then to such an (N, X, β) Fuller associates an index:

$$i(N, X, \beta) = \sum_{o \in N} \frac{1}{m(o)} i(o),$$

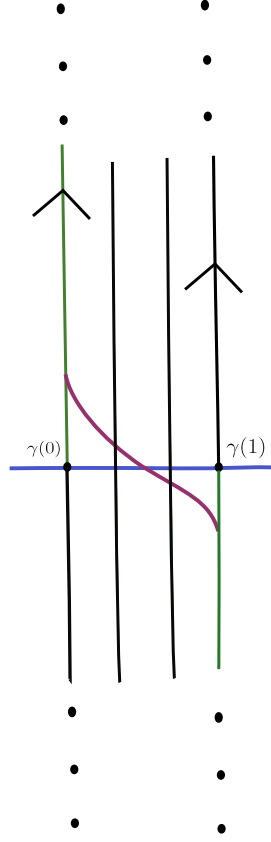


FIGURE 2. The green shaded path is γ , the indicated orientation is given by $u^*\lambda$, the D_1 foliation is shaded in black, the D_2 foliation is shaded in blue. The purple segment is part of the loop $o : S^1 \rightarrow \Sigma$, which is smooth and satisfies $\lambda(o'(t)) > 0$ for all t .

where $i(o)$ is the fixed point index of the time $p(o)$ return map of the flow of X with respect to a local surface of section in M transverse to the image of o .

Fuller then shows that $i(N, X, \beta)$ has the following invariance property. For a continuous homotopy $\{X_t\}$, $t \in [0, 1]$ set

$$S(\{X_t\}, \beta) = \{(o, t) \in L_\beta M \times [0, 1] \mid o \in S(X_t)\}.$$

And given a continuous homotopy $\{X_t\}$, $X_0 = X$, $t \in [0, 1]$, suppose that \tilde{N} is an open compact subset of $S(\{X_t\}, \beta)/S^1$, such that

$$\tilde{N} \cap (L_\beta M \times \{0\})/S^1 = N.$$

Then if

$$N_1 = \tilde{N} \cap (L_\beta M \times \{1\})/S^1$$

we have

$$i(N, X, \beta) = i(N_1, X_1, \beta).$$

In the case where X is the R^λ -Reeb vector field on a contact manifold (C^{2n+1}, λ) , and if o is non-degenerate, we have:

$$(A.2) \quad i(o) = \text{sign Det}(\text{Id} |_{\xi(x)} - F_{p(o),*}^\lambda |_{\xi(x)}) = (-1)^{CZ(o)-n},$$

where $F_{p(o),*}^\lambda$ is the differential at x of the time $p(o)$ flow map of R^λ , and where $CZ(o)$ is the Conley-Zehnder index, see [23].

B. REMARK ON MULTIPLICITY

This is a small note on how one deals with curves having non-trivial isotropy groups, in the virtual fundamental class technology. We primarily need this for the proof of Theorem 6.8.

Given a closed oriented orbifold X , with an orbundle E over X Fukaya-Ono [10] show how to construct using multi-sections its rational homology Euler class, which when X represents the moduli space of some stable curves, is the virtual moduli cycle $[X]^{vir}$. When this is in degree 0, the corresponding Gromov-Witten invariant is $\int_{[X]^{vir}} 1$. However, they assume that their orbifolds are effective. This assumption is not really necessary for the purpose of construction of the Euler class but is convenient for other technical reasons. A different approach to the virtual fundamental class which emphasizes branched manifolds is used by McDuff-Wehrheim, see for example McDuff [13], [16] which does not have the effectivity assumption, a similar use of branched manifolds appears in [6]. In the case of a non-effective orbundle $E \rightarrow X$ McDuff [15], constructs a homological Euler class $e(E)$ using multi-sections, which extends the construction [10]. McDuff shows that this class $e(E)$ is Poincare dual to the completely formally natural cohomological Euler class of E , constructed by other authors. In other words there is a natural notion of a homological Euler class of a possibly non-effective orbundle. We shall assume the following black box property of the virtual fundamental class technology.

Axiom B.1. *Suppose that the moduli space of stable maps is cleanly cut out, which means that it is represented by a (non-effective) orbifold X with an orbifold obstruction bundle E , that is the bundle over X of cokernel spaces of the linearized CR operators. Then the virtual fundamental class $[X]^{vir}$ coincides with $e(E)$.*

Given this axiom it does not matter to us which virtual moduli cycle technique we use. It is satisfied automatically by the construction of McDuff-Wehrheim, (at the moment in genus 0, but surely extending). It can be shown to be satisfied in the approach of John Pardon [22]. And it is satisfied by the construction of Fukaya-Oh-Ono-Ohta [8], the latter is communicated to me by Kaoru Ono. When X is 0-dimensional this does follow immediately by the construction in [10], taking any effective Kuranishi neighborhood at the isolated points of X , (this actually suffices for our paper.)

As a special case most relevant to us here, suppose we have a moduli space of elliptic curves in X , which is regular with expected dimension 0. Then its underlying space is a collection of oriented points. However, as some curves are multiply covered, and so have isotropy groups, we must treat this as a non-effective 0 dimensional oriented orbifold. The contribution of each curve $[u]$ to the Gromov-Witten invariant $\int_{[X]^{vir}} 1$ is $\frac{\pm 1}{|\Gamma([u])|}$, where $|\Gamma([u])|$ is the order of the isotropy group $\Gamma([u])$ of $[u]$, in the McDuff-Wehrheim setup this is explained in [13, Section 5]. In the setup of Fukaya-Ono [10] we may readily calculate to get the same thing taking any effective Kuranishi neighborhood at the isolated points of X .

8. ACKNOWLEDGEMENTS

This paper has undergone many approximations over a number of years, and so was influenced by quite a number of people. I want to thank in particular, Viktor Ginzburg for a crucial initial discussion, and Yasha Eliashberg for some encouragement.

REFERENCES

- [1] V. APOSTOLOV AND G. DLOUSSKY, *Locally conformally symplectic structures on compact non-Kähler complex surfaces*, Int. Math. Res. Not., 2016 (2016), pp. 2717–2747.
- [2] A. BANYAGA, *Some properties of locally conformal symplectic structures.*, Comment. Math. Helv., 77 (2002), pp. 383–398.
- [3] G. BAZZONI AND J. C. MARRERO, *On locally conformal symplectic manifolds of the first kind*, Bull. Sci. Math., 143 (2018), pp. 1–57.
- [4] F. BOURGEOIS, *A Morse-Bott approach to contact homology*, PhD thesis, (2002).

- [5] D. CANT, *Contactomorphisms of the sphere without translated points*, <https://arxiv.org/pdf/2210.11002.pdf>, (2022).
- [6] K. CIELIEBAK, I. MUNDET I RIERA, AND D. A. SALAMON, *Equivariant moduli problems, branched manifolds, and the Euler class.*, *Topology*, 42 (2003), pp. 641–700.
- [7] Y. ELIASHBERG AND E. MURPHY, *Making cobordisms symplectic*, *J. Am. Math. Soc.*, 36 (2023), pp. 1–29.
- [8] K. FUKAYA, Y.-G. OH, H. OHTA, AND K. ONO, *Technical details on Kuranishi structure and virtual fundamental chain*, arXiv.
- [9] K. FUKAYA, Y.-G. OH, H. OHTA, AND K. ONO, *Lagrangian Intersection Floer theory, Anomaly and Obstruction I and II*, AMS/IP, Studies in Advanced Mathematics, 2000.
- [10] K. FUKAYA AND K. ONO, *Arnold Conjecture and Gromov–Witten invariant*, *Topology*, 38 (1999), pp. 933 – 1048.
- [11] H.-C. LEE, *A kind of even-dimensional differential geometry and its application to exterior calculus.*, *Am. J. Math.*, 65 (1943), pp. 433–438.
- [12] M. MAZZUCHELLI, *Isometry-invariant geodesics and the fundamental group*, *Math. Ann.*, 362 (2015), pp. 265–280.
- [13] D. MCDUFF, *Notes on Kuranishi Atlases*, arXiv.
- [14] D. MCDUFF, *The local behaviour of holomorphic curves in almost complex 4-manifolds*, *J. Differ. Geom.*, 34 (1991), pp. 143–164.
- [15] ———, *Groupoids, branched manifolds and multisections.*, *J. Symplectic Geom.*, 4 (2006), pp. 259–315.
- [16] ———, *Constructing the virtual fundamental class of a Kuranishi atlas*, *Algebr. Geom. Topol.*, 19 (2019), pp. 151–238.
- [17] D. MCDUFF AND D. SALAMON, *Introduction to symplectic topology*, Oxford Math. Monographs, The Clarendon Oxford University Press, New York, second ed., 1998.
- [18] ———, *J-holomorphic curves and symplectic topology*, no. 52 in American Math. Society Colloquium Publ., Amer. Math. Soc., 2004.
- [19] M. J. MICALLEF AND B. WHITE, *The structure of branch points in minimal surfaces and in pseudoholomorphic curves*, *Ann. Math. (2)*, 141 (1995), pp. 35–85.
- [20] Y.-G. OH AND Y. SAVELYEV, *Pseudoholomorphic curves on the $\mathcal{L}\mathcal{C}\mathcal{G}$ -fication of contact manifolds*, *Adv. Geom.*, 23 (2023), pp. 153–190.
- [21] Y.-G. OH AND R. WANG, *Analysis of contact Cauchy-Riemann maps. I: A priori C^k estimates and asymptotic convergence*, *Osaka J. Math.*, 55 (2018), pp. 647–679.
- [22] J. PARDON, *An algebraic approach to virtual fundamental cycles on moduli spaces of J-holomorphic curves*, *Geometry and Topology*.
- [23] J. ROBBIN AND D. SALAMON, *The Maslov index for paths.*, *Topology*, 32 (1993), pp. 827–844.
- [24] S. SANDON, *A Morse estimate for translated points of contactomorphisms of spheres and projective spaces*, *Geom. Dedicata*, 165 (2013), pp. 95–110.
- [25] Y. SAVELYEV, *Extended Fuller index, sky catastrophes and the Seifert conjecture*, *International Journal of mathematics*, 29 (2018).
- [26] C. H. TAUBES, *The Seiberg-Witten equations and the Weinstein conjecture.*, *Geom. Topol.*, 11 (2007), pp. 2117–2202.
- [27] I. VAISMAN, *Locally conformal symplectic manifolds.*, *Int. J. Math. Math. Sci.*, 8 (1985), pp. 521–536.
- [28] C. WENDL AND C. GERIG, *Generic transversality for unbranched covers of closed pseudoholomorphic curves*, arXiv:1407.0678, (2014).
- [29] C. WENDLE, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four.*, *Comment. Math. Helv.*, 85 (2010), pp. 347–407.
- [30] ———, *Transversality and super-rigidity for multiply covered holomorphic curves*, arXiv:1609.09867, (2016).
- [31] A. ZINGER, *Notes on j-holomorphic maps*, arxiv:1706.00331, (2017).