A REMARK ON DEFORMATION OF GROMOV NON-SQUEEZING

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Abstract. Let R, r be as in the classical Gromov non-squeezing theorem, and let $\epsilon = (\pi R^2 - \pi r^2)/\pi r^2$. We first conjecture that the Gromov non-squeezing phenomenon persists for deformations of the symplectic form on the range C^0 (w.r.t. the standard metric) ϵ-nearby to the standard symplectic form. We prove this in some special cases, in particular when the dimension is four and when $R < \sqrt{2}r$. Given such a perturbation, we can no longer compactify the range and hence the classical Gromov argument breaks down. Our main method consists of a certain trap idea for holomorphic curves, analogous to traps in dynamical systems.

1. Introduction

One of the most important results to this day in symplectic geometry is the so called Gromov non-squeezing theorem, appearing in the seminal paper of Gromov [2]. Let $\omega_{st} = \sum_{i=1}^{n} dp_i \wedge dq_i$ denote the standard symplectic form on $\mathbb{R}^{2n}$. Gromov's theorem then says that there does not exist a symplectic embedding

$$(B_R, \omega_{st}) \hookrightarrow (S^2 \times \mathbb{R}^{2n-2}, \omega_{\pi r^2} \oplus \omega_{st}),$$

for $R > r$, with B_R the standard closed radius R ball in $\mathbb{R}^{2n}$ centered at 0, and $\omega_{\pi r^2}$ a symplectic form on S^2 with area πr^2. It is very natural to conjecture the following simple extension.

Conjecture 1. *Let $R > r > 0$ be given, set $\epsilon = (\pi R^2 - \pi r^2)/\pi r^2$ and let $\omega = \omega_{\pi r^2} \oplus \omega_{st}$ be the symplectic form on $M = S^2 \times \mathbb{R}^{2n-2}$ as above. Then for any symplectic form ω' on M, C^0 ϵ-close to ω, there is no symplectic embedding $\phi : (B_R, \omega_{st}) \hookrightarrow (M, \omega')$, meaning that $\phi^* \omega' = \omega_{st}$.* [1]

Here, the C^0 distance is with respect to the metric $g_J = \omega(\cdot, J\cdot)$ on M for J the standard complex structure, see (3.2). The above ϵ is of course optimal, for if $c > \epsilon$ there is a symplectic embedding of B_R into $(M, c \cdot \omega)$. It was pointed out to me by Spencer Cattalani that the conjecture fails if we replace S^2 by the radius r disk, by a very simple argument appearing in Gromov's original [2].

To prove this we cannot use the classical Gromov-Witten argument since we cannot compactify the range. Another idea is needed to get an appropriate compact moduli space of holomorphic curves. One possible approach is to use convexity or in other words the maximum principle. This approach can prove some cases of the conjecture but is very unlikely to yield a proof of the general case. This is because for a general ω' as above a compatible almost complex structure may be forced to be non-standard at infinity.

There is another approach that when sharpened should yield the general case of the conjecture, at least in dimension 4. This is based on a simple idea of holomorphic traps (Definition 2.1) somewhat analogous to traps in dynamical systems. We use this to prove some special cases of the conjecture.

Theorem 1.1. *When $n = 2$ the conjecture above holds in the following three cases:*

1. *$R < \sqrt{2}r$.*
2. *Let $p : S^2 \times \mathbb{R}^n \to \mathbb{R}^2$ be the projection map. There is a continuous deformation (topology as mentioned above) of symplectic forms $\{\omega_t\}$, $t \in [0, 1]$, $\omega' = \omega_1$, $\omega_0 = \omega$. And such that the*

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[1] After the publication of this note, Spencer Cattalani has found a counterexample in dimensions higher than 4, that is $n > 2$. We leave the conjecture here unmodified for consistency with the published version.

following is satisfied: there is a compact $K \subset \mathbb{R}^2$, such that for each $x \in \mathbb{R}^2 - K$, and each t, ω_t is non-degenerate on $p^{-1}(x)$.

(3) *There is a continuous deformation of symplectic forms $\{\omega_t\}$, $t \in [0,1]$, $\omega' = \omega_1$, $\omega_0 = \omega$ and a continuous family of projections $\{p_t : M \to \mathbb{R}^2\}$ s.t. for each t and $x \in \mathbb{R}^2$ ω_t is non-degenerate on $p_t^{-1}(x)$.*

2. A trap for holomorphic curves

For basic notions of J-holomorphic curves we refer the reader to [6].

Definition 2.1. *Let (M, J) be an almost complex manifold, and $A \in H_2(M)$ fixed. Let $K \subset M$ be a closed subset. Suppose that for every $x \in \partial K$ (the topological boundary) there is a J-holomorphic, real codimension 2, compact submanifold $H_x \ni x$ of M, satisfying:*

- *$H_x \subset K$.*
- *$A \cdot H_x \leq 0$, where the left-hand side is the homological intersection number.*

*We call such a K a **J-holomorphic trap** (for class A curves).*

Lemma 2.2. *Let M, J and A be as above, and K be a J-holomorphic trap for class A curves. Let $u : \Sigma \to M$ be a J-holomorphic curve in class A, with Σ a connected closed Riemann surface. Then*

$$(\operatorname{image} u \cap K) \neq \emptyset \implies \operatorname{image} u \subset K.$$

Proof. Suppose that u intersects ∂K, otherwise we already have $\operatorname{image} u \subset \mathit{interior}(K)$, since $\operatorname{image} u$ is connected (and by elementary topology). Then u intersects H_x as in the definition of a holomorphic trap, for some x. Consequently, as $A \cdot H_x \leq 0$, by positivity of intersections [6, Section 2.6], $\operatorname{image} u \subset H_x \subset K$. □

3. Proof of Theorem 1.1

Definition 3.1. *A pair (ω, J) of a 2-form ω on a smooth manifold M and an almost complex structure J on M are* **compatible** *if $\omega(\cdot, J\cdot)$ defines a J-invariant inner product $g_{\omega,J}$ on M.*

Let us quickly recall the definition of the C^0 distance d_{C^0}, on the set of 2-forms $\Omega^2(M)$ for a fixed metric g on M.

$$d_{C^0}(\omega_0, \omega_1) = \sup_{|v \wedge w|_g = 1} |\omega_0(v,w) - \omega_1(v,w)|, \tag{3.2}$$

where more specifically, the supremum is over all g-norm 1 simple bivectors $v \wedge w$ in $\Lambda^2(TM)$.

Let ω be the symplectic form on $M = S^2 \times \mathbb{R}^2$ as in the statement of Conjecture 1. In our case d_{C^0} will be defined with respect to the metric $g_{\omega,J}$ as in Definition 3.1 for J the standard product complex structure.

We now prove the second case of the theorem. Let $\epsilon = (\pi R^2 - \pi r^2)/\pi r^2$. Suppose by contradiction that there is a d_{C^0}-continuous family $\{\omega_t\}$ of symplectic forms s.t.

- $d_{C^0}(\omega, \omega_1) < \epsilon$.
- There exists a symplectic embedding
$$\phi : (B_R, w_{st}) \hookrightarrow (M, \omega_1).$$
- For each $t \in [0,1]$, ω_t is non-degenerate on the fibers M_x of the projection
$$p : M \to \mathbb{R}^2,$$
for $x \in \mathbb{R}^2 - K'$ for some compact $K' \subset \mathbb{R}^2$.

Set $B := \phi(B_R)$ and let $D^\circ \supset (p(B) \cup K')$ be an open standard disk in $\mathbb{R}^2$, and let D denote its closure. So $K = S^2 \times D$ is a compact subset of M, with the properties:

(1) ∂K is smoothly foliated by the fibers M_x.
(2) For each t, ω_t is non-degenerate on the fibers M_x contained in ∂K.

We denote by $T^{vert}\partial K \subset TM$, the sub-bundle of vectors tangent to the leaves of the above-mentioned foliation.

Let j be the standard complex structure on B_R. We may extend $\phi_* j$ to an ω_1-compatible almost complex structure J_1 on M, preserving $T^{vert}\partial K$ using:

- image ϕ does not intersect ∂K.
- The non-degeneracy of ω_1 on the fibers.
- The well known existence/flexibility results for compatible almost complex structures on symplectic vector bundles, see for instance [5, Section 2.6].

We may then extend J_1 to an appropriately [2] smooth family $\{J_t\}$, $t \in [0,1]$, of almost complex structures on M, s.t. J_t is ω_t-compatible for each t, with $J_0 = J$ as above, and such that J_t preserves $T^{vert}\partial K$ for each t. The latter condition can be satisfied by similar reasoning as above, using that ω_t is non-degenerate on the fibers M_x, contained in ∂K for each t.

Such fibers are J_t-holomorphic hypersurfaces for each t, and smoothly foliate ∂K. Moreover, if $A = [S^2] \otimes [pt] \in H_2(M)$ then the intersection number of A with a fiber is 0. That is $A \cdot p^{-1}(z) = 0$, for $\forall z \in \mathbb{R}^2$. And so K is a compact J_t-holomorphic trap for class A curves, for each t.

Set $x_0 := \phi(0)$. Denote by $\mathcal{M}_t$ the space of equivalence classes of maps $u : \mathbb{CP}^1 \to M$, where u is a J_t-holomorphic, class A curve passing through x_0. The equivalence relation is by the usual biholomorphism reparametrization group action, so that $u \sim u'$ if there exists a biholomorphism $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ s.t. $u' = u \circ f$. Then $\mathcal{M} = \cup_t \mathcal{M}_t$ is compact by energy minimality of A (which rules out bubbling), by Lemma 2.2, and by compactness of K.

To use Gromov-Witten type curve counts, we need to regularize. We may use the "standard" Banach approach. This has the advantage of being readily understood by experts but a possible disadvantage of appearing opaque and ad hoc to new-comers to the field. For this reason we will also give an independent argument using polyfold theory.

3.1. **Banach approach.** This is based on [6] and the picture is as follows. Let $\mathcal{B}$ be the universal Banach moduli space of class A curves:

$$\mathcal{B} = \mathcal{M}^*(A, \mathcal{J}^l) := \{(u, J) \mid J \in \mathcal{J}^l, u : \mathbb{CP}^1 \to M \text{ is a simple class } A \ J\text{-holomorphic curve}\},$$

where $\mathcal{J}^l$ is the space of class C^l almost complex structures, taking l to be sufficiently large. Then we have an evaluation map $ev : \mathcal{B} \to M$, $(u, J) \mapsto u(z_0)$, for $z_0 \in \mathbb{CP}^1$ fixed. Let $\pi : \mathcal{B} \to \mathcal{J}^l$ be the projection.

The product map

$$\mathcal{B} \xrightarrow{ev \times \pi} M \times \mathcal{J}^l$$

is a Fredholm map. There is one immediate problem: given $(x_0, J) \in M \times \mathcal{J}^l$ a priori we may not be able to perturb it to a regular value of the form (x', J') (that is we may need to perturb x_0 to x'). This would complicate the last step of the proof of the theorem, which needs specifically a holomorphic curve through x_0. Fortuitously, it turns out that the map ev is always a submersion, see [Proposition 3.4.2] [6]. Thus, there is no need to perturb x_0.

Lemma 3.3. *Let $\{J_t\}$, $t \in [0,1]$, be the family as constructed above. Then there is a path $p' : [0,1] \to M \times \mathcal{J}^l$, $t \mapsto (x_0, J'_t)$, such that:*

1. *$ev \times \pi$ is transverse to p' in the standard differential topology sense, (this is equivalent to $\{J'_t\}$ being a regular homotopy, as defined in [Definition 3.1.7][6]).*
2. *J'_t is ω_t-compatible for each t.*
3. *J'_t preserves $T^{vert}\partial K$ for each t.*
4. *$J'_0 = J$.*

[2]Because M is not compact, we need to treat the space of almost complex structures as a nuclear LF manifold, [7] rather than a Frechet manifold. However, this is only cosmetic (in the setup of the moment) since we are only interested in the behavior over a fixed compact set. Thus, we could also use a modified Frechet topology induced by choosing a fixed compact.

Proof. Only condition (3) requires an explanation. To see that this can be satisfied, take open $U_1, U_2 \subset M$, homeomorphic to an open ball in $\mathbb{R}^4$, with $\overline{U}_1 \subset U_2$ with $B \subset U_1$ and with $U_2 \subset K$. Now any closed J-holomorphic curve which intersects U_1 must intersect $U_2 - \overline{U}_1$. As otherwise we contradict $H_2(U_1, \mathbb{Z}) = 0$, since the homology class of closed, non-constant, J-holomorphic curve is never zero, for J compatible with a symplectic form. Thus, the family $\{J_t\}$ can be be regularized by perturbing only within the region $U_2 - \overline{U}_1$ cf. [6, proof of Lemma 3.4.4]. In this case, (3) will be automatically satisfied. □

For p' as in the lemma, define $\mathcal{M}'$ to be the preimage $(ev \times \pi)^{-1}(\text{image}\, p')/\sim$, where $\sim$ is the following equivalence relation. $u \sim u'$ if there is a biholomorphism $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ s.t. $u' = u \circ f$ and s.t. $f(z_0) = z_0$. Then $\mathcal{M}'$ is a compact one dimensional manifold.

The boundary component $(ev \times \pi)^{-1}(x_0, J)/\sim$ is a point corresponding to the single J-holomorphic, class A curve passing through x_0. It follows that the boundary component $(ev \times \pi)^{-1}(x_0, J')/\sim$ is likewise non-empty. Then let $u_0 \in (ev \times \pi)^{-1}(x_0, J')/\sim$, we will use this further ahead.

3.2. **Polyfold approach.** Alternatively, we may use Hofer-Wysocki-Zehnder polyfold regularization in Gromov-Witten theory, especially as recently worked out in this present context by the team of Franziska Beckschulte, Ipsita Datta, Irene Seifert, Anna-Maria Vocke, and Katrin Wehrheim. We can also of course use other virtual approaches, but this is not instantaneous, for example if we were to invoke Pardon [8] then we would have needed to construct implicit atlases in the constrained case (this can be done of course).

As explained in [1, Section 3.5], in a essentially identical situation, we may embed $\mathcal{M}$ into a natural polyfold setup of Hofer-Wysocki-Zehnder [4]. More to the point, we express $\mathcal{M}$ as the zero set of an sc-Fredholm section of a suitable (tame, strong) M-polyfold bundle. The only difference with the setup of [1, Section 3.5] is that they compactify M to $S^2 \times T^2$, to get a compact moduli space. We of course cannot compactify, but remember that we used the holomorphic trap idea to force compactness of $\mathcal{M}$. And so we are in an equivalent situation.

Again as in [1], we take the M-polyfold regularization of $\mathcal{M}$. This gives a one dimensional compact cobordism $\mathcal{M}^{reg}$ between $\mathcal{M}^{reg}_0$ and $\mathcal{M}^{reg}_1$.

Now $\mathcal{M}^{reg}_0$ is a point: corresponding to the unique $(J = J_0)$-holomorphic class A, curve $u : \mathbb{CP}^1 \to M$ passing through x_0. Consequently, $\mathcal{M}^{reg}_1$ is non-empty, that is there is a J_1-holomorphic class A curve $u_0 : \mathbb{CP}^1 \to M$ passing through x_0.

3.3. **Finishing the proof.** Now $\langle \omega, A \rangle = \pi \cdot r^2$, and we have a representative for A whose g-area is πr^2. So we have:

$$|\langle \omega_1, A \rangle - \pi \cdot r^2| = |\langle \omega_1, A \rangle - \langle \omega, A \rangle| < \epsilon \pi \cdot r^2 = \pi R^2 - \pi r^2,$$

where the inequality uses that $d_{C^0}(\omega, \omega_1) < \epsilon$. So we get

$$|\int_{\mathbb{CP}^1} u_0^* \omega_1 - \pi r^2| < \pi R^2 - \pi r^2.$$

And consequently,

$$\int_{\mathbb{CP}^1} u_0^* \omega_1 < \pi R^2.$$

We may then proceed exactly as in the now classical proof of Gromov [3] of the non-squeezing theorem to get a contradiction and finish the proof. More specifically, $\phi^{-1}(\text{image}\, \phi \cap \text{image}\, u_0)$ is a minimal surface in B_R, with boundary on the boundary of B_R, and passing through $0 \in B_R$. By construction it has area strictly less than πR^2, which is impossible by the classical monotonicity theorem of differential geometry. See also [1, Lemma A.2] where the monotonicity theorem is suitably generalized, to better fit the present context.

This finishes the proof of the second case. To prove the first case, note that if v is a g-unit vector then $\omega(v, Jv) = 1$. If $\epsilon = (\pi R^2 - \pi r^2)/\pi r^2$ and $\pi R^2 < 2\pi r^2$ then $\epsilon < 1$. And so if ω' is ϵ close to ω then $\omega'(v, Jv) > 0$. It follows that:

- $\omega_t = (1-t)\omega + t\omega'$ is non-degenerate, for each $t \in [0, 1]$.

- For each $t \in [0,1]$, ω_t is non-degenerate on the fibers M_x of the natural projection
$$p : (M = S^2 \times \mathbb{R}^2) \to \mathbb{R}^2,$$
for all $x \in \mathbb{R}^2$.

So the family $\{\omega_t\}$ satisfies the hypothesis of the second case taking $K = \emptyset$, and the conclusion follows.

For case 3, suppose we are given such a family of projections p_t, and let $D \subset \mathbb{R}^2$, be a closed disk constructed as in the proof of case 2. Define $\widetilde{p} : M \times [0,1] \to \mathbb{R}^2$ by $\widetilde{p}(x,t) = p_t(x)$, then by assumptions $\widetilde{p}$ is continuous. Define the compact subset of $M \times [0,1]$:
$$K_D = \widetilde{p}^{-1}(D).$$
As in the proof of case 2, we get a family $\{J_t\}$ of ω_t compatible almost complex structures, s.t. K_D is trapping for corresponding cobordism moduli space. That is for each $(u,t) \in \mathcal{M}$ the image of u is contained in $K_D \cap M \times \{t\}$. It follows that $\mathcal{M}$ is compact, then proceed as in the proof of case 2. □

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