SPECTRAL GEOMETRY OF THE GROUP OF HAMILTONIAN SYMPLECTOMORPHISMS

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ABSTRACT. We introduce here a natural functional associated to any $b \in QH_*(M,\omega)$: *spectral length functional*, on the space of "generalized paths" in $\text{Ham}(M,\omega)$, closely related to both the Hofer length functional and spectral invariants and establish some of its properties. This functional is smooth on its domain of definition, and moreover the nature of extremals of this functional suggests that it may be variationally complete, in the sense that any suitably generic element of $\widetilde{\text{Ham}}(M,\omega)$ is connected to *id* by a generalized path minimizing spectral length. Rather strong evidence is given for this when $M = S^2$, where we show that all the Lalonde-McDuff Hamiltonian symplectomorphisms are joined to id by such a path. We also prove that the associated norm on $\text{Ham}(M,\omega)$ is non-degenerate and bounded from below by the the spectral norm. If the spectral length functional is variationally complete the associated norm reduces to the spectral norm.

1. INTRODUCTION

The Hofer length functional and the resulting Finsler geometry on $\text{Ham}(M,\omega)$: the Hofer geometry, has been one of the driving forces in symplectic geometry. It provides a direct link from the world of symplectic topology to the world of metric spaces. This is combined with its naturality, simplicity of definition, and intriguing connections with "hard" symplectic geometry: for example in the sense of elliptic methods of Gromov and Floer.

However this functional has a disturbing variation incompleteness problem, as Lalonde-McDuff [2] observed that in $\text{Ham}(S^2,\omega)$, there are elements not joined by locally (in the path space) length minimizing Hofer geodesics, i.e. stable geodesics. For future reference we shall refer to these elements by **Lalonde-McDuff symplectomorphisms**. This is one of the problems stalling progress in understanding global metric properties of $\text{Ham}(M,\omega)$.

The main goal here is to describe another highly natural functional L^s_j, depending on an almost complex structure j on M, on a certain "generalized path space" of $\text{Ham}(M,\omega)$. This functional is no longer so elementary in nature and is deeply intertwined with chain level Floer theory, and spectral invariants. It may seem awkward that L^s_j depends on j, but surprisingly extremals for L^s_j and their lengths are independent of j. Consequently if L^s_j was suitably variationally complete the associated distance function would depend only on ω, and in this case we show that the associated spectral distance from *id* to $\widetilde{\phi} \in \widetilde{\text{Ham}}(M,\omega)$ reduces to the spectral norm of $\widetilde{\phi}$.

Here is the main criterion for being L^s_j minimizing that is used in this paper.

Proposition 1.0.1. *Suppose a pair of regular $\widetilde{\phi}_\pm \in \widetilde{Ham}(M,\omega)$ can be connected by a strong Cerf homotopy in degree 2n, then there is an L^s_j minimizing generalized path between $\widetilde{\phi}_\pm$.*

A strong Cerf homotopy $H_r : M \times S^1 \to \mathbb{R}$, $0 \leq r \leq 1$, in degree 2n, is roughly speaking a Cerf-Floer homotopy with nice behavior in CZ degree 2n. Details are given in Section 3. To state our main results we need to give the full background.

1.1. **Spectral length functional.**

1.1.1. *Hamiltonian category $\mathcal{G}_M$.* Let (M,ω) be a closed symplectic manifold. To this we associate a topological category whose objects are lifts to the universal cover of Hamiltonian symplectomorphisms of M and the "simple" morphism space $\mathcal{P}^s_\pm$ from $\widetilde{\phi}_-$ to $\widetilde{\phi}_+$ consists of Hamiltonian connections $\mathcal{A}$ on the topologically trivial bundle

$$\pi : P = M \times (\mathbb{R} \times S^1) \to \mathbb{R} \times S^1,$$

with the property that $\mathcal{A}$ over $r \times S^1 \subset \mathbb{R} \times S^1$ stabilizes to $\mathcal{A}_-$, respectively $\mathcal{A}_+$ for sufficiently small, respectively large r, and the holonomy of $\mathcal{A}_\pm$ is $\widetilde{\phi}_\pm \in \widetilde{\text{Ham}}(M,\omega)$. Also for r sufficiently small, respectively large $\mathcal{A}$ is assumed to be flat in the r direction, i.e. the horizontal lift of $\frac{\partial}{\partial r}|_z$ is $\frac{\partial}{\partial r}|_{x,z}$, (and consequently $\mathcal{A}$ is flat outside a compact region of P by above assumptions). The compact region of P, where $\mathcal{A}$ is not assumed to be r-flat, will be called *principal*.

Remark 1.1.1. *We should be able to relax the notion of morphism to allow $\mathcal{A}$, which are only asymptotic to flat connections as $r \hookrightarrow \pm\infty$, converging sufficiently fast. All of the discussion in this paper should go through for this case. We have no use for such morphisms here, but they may potentially be useful when studying the question of variational completeness more closely. We should also be able to take higher genus Riemann surfaces with boundary instead of $\mathbb{R} \times S^1$ and take tuples $\{\widetilde{\phi}_i\}$ of lifts of Hamiltonian symplectomorphisms as objects, analogously to* [3].

We then partially compactify this space by allowing SFT style degenerations of the morphisms. The full morphism space $\mathcal{P}(\widetilde{\phi}_-, \widetilde{\phi}_+)$ often abbreviated $\mathcal{P}_\pm$, then consists of simple morphisms and formal concatenations of simple morphisms. The identity morphism $id_{\widetilde{\phi}}$ in this category is just an r-flat connection $\mathcal{A}_{\widetilde{\phi}}$, with holonomy over $r \times S^1$ being $\widetilde{\phi}$ for every r. For this to be a true category we need to put a mild equivalence relation, which we suppress. (More appropriately this is a 2-category.)

1.1.2. *From paths in $Ham(M,\omega)$ to morphisms of $\mathcal{G}_M$.* Let

$$p : [0,1] \to \text{Ham}(M,\omega),$$

$p(0) = id$, be generated by $H : M \times [0,1] \to \mathbb{R}$. To such a p there is a naturally associated Hamiltonian connection $\mathcal{A}_p$ on $P \to \mathbb{R} \times S^1$, which is a morphism from id to $p(1)$ in $\mathcal{G}_M$. Explicitly $\mathcal{A}_p$ is constructed as follows: for $0 \leq r \leq 1$ the horizontal lift of $\frac{\partial}{\partial \theta} \in T_{r,\theta}(\mathbb{R} \times S^1)$ is determined by the condition that the holonomy path ϕ_θ, $\theta \in S^1$, of $\mathcal{A}|_{r \times S^1}$ is generated by $\eta(r) \cdot H$, while being flat in the r direction, where $\eta : \mathbb{R} \to \mathbb{R}$ is a fixed function satisfying:

$$\eta(r) = \begin{cases} 1 & \text{if } 1-\delta \leq r \leq 1, \\ r & \text{if } 2\delta \leq r \leq 1-2\delta, \\ 0 & \text{if } r \leq \delta \end{cases} \tag{1.1}$$

for a small $\delta > 0$.

1.1.3. *Floer chain complex.* For more details on Floer homology and Hamiltonian perturbations the reader may see for example [5, Chapter 8, 12]. This is a chain complex associated to a $\widetilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$. We fix an ω-compatible complex structure j on M once and for all. For the definition of the chain complex we need a suitably generic Hamiltonian connection $\mathcal{A}_{\widetilde{\phi}}$ on $M \times S^1$ with holonomy $\widetilde{\phi}$, such a connection will be called Morse-Floer. Then $CF_*(\mathcal{A}_{\widetilde{\phi}})$ is generated over $\mathbb{Q}$ by pairs $\widetilde{\gamma} = (\gamma, \bar{\gamma})$, with γ a flat section of $\mathcal{A}_{\widetilde{\phi}}$ and $\bar{\gamma}$ an equivalence class of a bounding section in $M \times D^2$, with two bounding disks considered equivalent if their difference (as 2-chains) is annihilated by $c_1(TM)$. The grading is given by Conley-Zehnder index. The boundary operator is defined through count of $J_{\mathcal{A}_{\widetilde{\phi}}}$-holomorphic sections of P, whose projections to $M \times S^1$ are asymptotic in backward, forward time to generators of $CF_*(\mathcal{A}_{\widetilde{\phi}})$. Here $J_{\mathcal{A}_{\widetilde{\phi}}}$ is the complex structure induced by the extension of the connection $\mathcal{A}_{\widetilde{\phi}}$ to a connection on P, flat in the r direction, more details are given below.

1.1.4. *From morphisms of $\mathcal{G}$ to almost complex structures on P.* Given $\mathcal{A} \in \mathcal{P}_{\pm}$ there is a *π-compatible* almost complex structure $J_{\mathcal{A}}$ on P induced by $\mathcal{A}$, which means the following:

- The natural map $\pi : (P, J_{\mathcal{A}}) \to (\mathbb{R} \times S^1, j)$ is holomorphic.
- $J_{\mathcal{A}}$ preserves the horizontal subbundle $Hor^{\mathcal{A}}$ of TX induced by $\mathcal{A}$.
- $J_{\mathcal{A}}$ preserves the vertical tangent bundle of $M \hookrightarrow P \to \mathbb{R} \times S^1$, and restricts to the fixed complex structure j on M.

Notation 1.1. *To cut down on notation we will sometimes forgo notationally distinguishing between $\phi \in Ham(M, \omega)$ and its lifts to universal cover.*

1.1.5. *Definition of the spectral length functional.* Let (M, ω) be a closed monotone symplectic manifold with a positive monotonicity constant and let $\mathcal{A}_{\pm} = \mathcal{A}_{\phi_{\pm}}$ be Morse-Floer. Let $\{\widetilde{\gamma}_i^-\}$, $\{\widetilde{\gamma}_j^+\}$ be a collection of natural geometric generators for $CF_k(\mathcal{A}_-)$, respectively $CF_k(\mathcal{A}_+)$ over $\mathbb{Z}$, and let $\mathcal{M}(\widetilde{\gamma}_i^-, \widetilde{\gamma}_j^+)$ denote the moduli space (whose virtual dimension is 0 by the index theorem) of holomorphic sections of P asymptotic to $\widetilde{\gamma}_i^-$, $\widetilde{\gamma}_j^+$.

Now let $K_{\mathcal{A}} : \mathbb{R} \times S^1 \to C^\infty(M)$ denote the Hodge star of the Lie algebra valued curvature form $R_{\mathcal{A}}$ of $\mathcal{A}$, with respect to standard Kahler metric g_j on $\mathbb{R} \times S^1$. In other words $K_{\mathcal{A}}(z) \in C^\infty(M)$ is the Lie algebra element $R_{\mathcal{A}}(v, w)$, for v, w an orthonormal pair at $z \in \mathbb{R} \times S^1$. However, we will instead think of $K_{\mathcal{A}}$ in terms of the naturally associated function $K_{\mathcal{A}} : P \to \mathbb{R}$. We now define a variant of the continuation map:

$$\Psi^k_E : CF_k(\mathcal{A}_-) \to CF_k(\mathcal{A}_+), \tag{1.2}$$

$$\Psi^k_E(\widetilde{\gamma}^-_i) = \sum_j \#\mathcal{M}(\widetilde{\gamma}^-_i, \widetilde{\gamma}^+_j)_E \, \widetilde{\gamma}^+_j, \tag{1.3}$$

where $\mathcal{M}(\widetilde{\gamma}^-_i, \widetilde{\gamma}^+_j)_E$ is the space of $J_{\mathcal{A}}$-holomorphic sections u which satisfy

$$\int_u K_{\mathcal{A}} \, \pi^* \omega_{st} \leq E, \tag{1.4}$$

with ω_{st} denoting the standard area form on $\mathbb{R} \times S^1$.

Definition 1.1.2. *For a degree k element $b \in QH_*(M)$ we define*

$L^s_j(\mathcal{A}, b) = \inf\{E | \textit{s.t. } PSS(b) \in FH_*(\mathcal{A}_+)$ *can be represented by a chain in* image $\Psi^k_E\}$.

Note that by our assumptions on $\mathcal{A}$, $K_{\mathcal{A}}$ vanishes outside a compact subset of P so that the integral (1.4) is finite. When b is the fundamental class we abbreviate $L^s_j(\mathcal{A}, [M])$ by $L^+_j(\mathcal{A})$. When $\widetilde{\phi}_- = id$, we define $L^+_j(\mathcal{A})$ naturally, by dropping constraints on the left in the definition of Ψ^k_E, i.e. we just count elements in $\mathcal{M}(\widetilde{\gamma}^+_j)$. This will be an important special case.

Our assumption that M, ω is monotone with positive monotonicity constant insures that only finitely many holomorphic sections u come into the definition of Ψ^k_E, and consequently L^s_j is a priori defined and is *smooth* on an open dense subspace of $\mathcal{P}_\pm$, consisting of *regular* $\mathcal{A}$, i.e. those for which the associated CR operator is surjective for each $u \in \mathcal{M}(\widetilde{\gamma}^-_i, \widetilde{\gamma}^+_j)$.

Let $\overline{\mathcal{A}} = (id \times \sigma)^* \mathcal{A}$, where

$$id \times \sigma : M \times (\mathbb{R} \times S^1) \to M \times (\mathbb{R} \times S^1), \tag{1.5}$$

and where σ is an orientation preserving reflection. Then $\overline{\mathcal{A}}$ is a morphism from ϕ^{-1}_+ to ϕ^{-1}_-. We set $L^-_j(\mathcal{A}) = L^s_j(\overline{\mathcal{A}}, [M])$. Define

$$L^s_j(\mathcal{A}) = L^+_j(\mathcal{A}) - L^-_j(\mathcal{A}).$$

and define a function $\widetilde{\text{Ham}}(M, \omega) \to \mathbb{R}$:

$$\widetilde{\phi} \mapsto |\widetilde{\phi}|_j \equiv \inf_{\mathcal{A} \in \mathcal{P}(id, \widetilde{\phi})} |L^s_j(\mathcal{A})|,$$

We will use the same notation for the pushdown of this function to $\text{Ham}(M, \omega)$, i.e.

$$|\phi|_j = \inf\{|\widetilde{\phi}|_j \text{ s.t. } \widetilde{\phi} \text{ projects to } \phi\}.$$

1.2. **Main results.**

Proposition 1.2.1. *The "norm" $|\cdot|_j$ is non-degenerate. Furthermore*

$$L^s_j(\mathcal{A}) \geq |\widetilde{\phi}_+|_s - |\widetilde{\phi}_-|_s,$$

where $|\widetilde{\phi}|_s$ is the usual spectral pseudo norm of Oh-Schwarz on the universal cover.

Definition 1.2.2. *We call $\mathcal{A}$ quasi-flat if $L^s_j(\mathcal{A}) = |\widetilde{\phi}_+|_s - |\widetilde{\phi}_-|_s$.*

By above proposition, such $\mathcal{A}$ are L^s_j minimizing on $\mathcal{P}_\pm$. The name is due to fact that such $\mathcal{A}$ have certain distinguished flat sections, which will be apparent from the definition of spectral length to be given later. This condition could also be viewed as relaxation of the condition for being L^+ minimizing, which translated into language of connections requires that $\mathcal{A}$ has a special, constant flat section.

Inspired by this we conjecture:

Conjecture 1.2.3. *For $\widetilde{\phi} \in \widetilde{Ham}(M, \omega)$ Floer non-degenerate, L^s_j attains a minimum on the generalized path space $\mathcal{P}(id, \widetilde{\phi})$. Moreover, this minimum can be represented by a quasi-flat morphism, and so*

$$|\widetilde{\phi}|_j = |\widetilde{\phi}|_s.$$

As a step towards this conjecture we have:

Theorem 1.2.4. *The class of Lalonde-McDuff Hamiltonian symplectomorphisms of S^2, can be joined to id by quasi-flat and hence L^s_j minimizing morphisms.*

Consider the direct analogue of the Hofer length functional defined on $\mathcal{P}_\pm$ by

$$L_H(\mathcal{A}) = \int_{\mathbb{R}\times S^1} \max_M K_{\mathcal{A}}(z) - \min_M K_{\mathcal{A}}(z)\, \omega_{st}.$$

Here is another corollary:

Corollary 1.2.5. *The distance function d_H induced by L_H gives a non-degenerate norm on $Ham(M, \omega)$, bounded from below by the spectral norm.*

Proof. We clearly have $L_H(\mathcal{A}) \geq L^s_j(\mathcal{A})$ and so the corollary follows. □

1.3. **Applications to Hofer geometry.** It may be difficult to see how the above theory can be useful in classical Hofer geometry. In fact the applications we have in mind require another conjecture, which we now give. First a definition:

Definition 1.3.1. *Define the* **pseudo-injectivity** *radius of a symplectic manifold M, ω as*

$$inj(M, \omega) = \sup\{r|\ s.t.\ |\widetilde{\phi}|^s_j < r \Rightarrow \text{ space of quasi-flat morhispms in } \mathcal{P}(id, \widetilde{\phi}) \text{ is non-empty and contractible.}\}$$

Note that this quantity is j independent.

Conjecture 1.3.2.

$$inj(S^2, \omega_{st}) = 1,\ \text{where} \int_{S^2} \omega_{st} = 1.$$

Moreover the space fibering over the Hofer $(1-\epsilon)$-ball in $\widetilde{Ham}(S^2, \omega_{st})$, $B(1-\epsilon)$ with fiber over $\phi \in B(1-\epsilon)$: the space of quasi-flat morphisms from id to ϕ, is a Serre fibration. (For any $\epsilon > 0$.)

Given this we may immediately deduce contractability in C^∞ topology of Hofer $(1-\epsilon)$-ball in $\mathrm{Ham}(S^2, \omega)$, which by itself is an open question and its solution opens the door on other interesting open problems.

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2. Properties of the spectral length functional

2.1. **Some notations and conventions.** The action functional

$$A_H : \widetilde{\mathcal{L}M} \to \mathbb{R},$$

is defined by

$$A_H(\gamma, D) = -\langle \omega, D \rangle + \int_0^1 H(\gamma(t), t) dt,$$

where $H : M \times S^1 \to \mathbb{R}$. The induced Hamiltonian flow X_t is given by

$$\omega(X_t, \cdot) = -dH_t(\cdot).$$

The positive Hofer length functional $L^+(p)$, for a path p in $\text{Ham}(M, \omega)$ is defined by

$$L^+(p) = \int_0^1 \max H_t \; dt,$$

where H_t is the generating Hamiltonian for p, normalized by condition

$$\int_M H_t \cdot \omega^n = 0. \tag{2.1}$$

While the Hofer length functional L is defined by

$$L(p) = \int_0^1 \max H_t - \min H_t \; dt.$$

The Conley-Zehnder index is normalized so that for a C^2-small Morse function $H_t = H$, the orbits corresponding to critical points (with trivial bounding disks) have CZ equal to Morse index. For future reference we note that under above condition, the CZ index of a constant orbit (x, D) for x a critical point and D an equivalence class of a bounding disk is the Morse index of x minus $2\langle c_1(TM), D \rangle$.

Proof of Proposition 1.2.1. From $\mathcal{A}$ we may construct a certain remarkable closed 2-form $\Omega_{\mathcal{A}}$, called the coupling form, originally constructed in [1]. This form is characterized as follows, the restriction of $\Omega_{\mathcal{A}}$ to fibers M of $P \to \mathbb{C}$ coincides with ω, the $\Omega_{\mathcal{A}}$-orthogonal subspaces in TP to fibers M are the horizontal subspaces and the value of $\Omega_{\mathcal{A}}$ on horizontal lifts of an orthonormal pair $\widetilde{v}, \widetilde{w} \in T_{m,z}P$ of $v, w \in T_z(\mathbb{C})$ is given by

$$\Omega_{\mathcal{A}}(\widetilde{v}, \widetilde{w}) = -K_{\mathcal{A}}(m, z), \tag{2.2}$$

for $m \in M_z$. For the full construction the reader is referred to [4, Section 6.4]. The most important property of $\Omega_{\mathcal{A}}$ is that its integral over a section of P asymptotic to $\widetilde{\gamma}_-, \widetilde{\gamma}_+$, is just

$$-\,(A_{H_+}(\widetilde{\gamma}_+) - A_{H_-}(\widetilde{\gamma}_-)). \tag{2.3}$$

There exists some $u \in \mathcal{M}(\mathcal{A}, \widetilde{\gamma}_-, \widetilde{\gamma}_+)$, for some $\widetilde{\gamma}_\pm$ with $A_{H_+}(\widetilde{\gamma}_+) = \rho(\widetilde{\phi}_+, [M])$ and $A_{H_-}(\widetilde{\gamma}_-) = \rho(\widetilde{\phi}_-, [M])$. And let $\overline{u}$ be some completely analogous section corresponding to $\overline{\mathcal{A}}$, where $\overline{\mathcal{A}}$ is as in (1.5). Then we have

$$0 \le \int_u \Omega_{\mathcal{A}} + \int_u K_{\mathcal{A}} \pi^* \omega_{st}, \tag{2.4}$$

$$0 \le \int_{\overline{u}} \Omega_{\overline{\mathcal{A}}} + \int_{\overline{u}} K_{\overline{\mathcal{A}}} \pi^* \omega_{st}, \tag{2.5}$$

By (2.3) we get

$$-\int_u \Omega_{\mathcal{A}} = \rho(\phi_+, [M]) - \rho(\phi_-, [M]),$$

where $\rho(\phi, [M])$ is the usual spectral invariant of Oh-Schwarz. Similarly,

$$-\int_{\overline{u}} \Omega_{\overline{\mathcal{A}}} = -(\rho(\phi_+^{-1}, [M]) - \rho(\phi_-^{-1}, [M])).$$

Consequently, subtracting (2.4), (2.5) we get

$$\rho(\phi_+, [M]) + \rho(\phi_+^{-1}, [M]) - \rho(\phi_-, [M]) - \rho(\phi_-^{-1}, [M]) \leq L_j^+(\mathcal{A}) - L_j^-(\mathcal{A}),$$

i.e.

$$|\phi_+|_s - |\phi_-|_s \leq L_j^+(\mathcal{A}) - L_j^-(\mathcal{A}).$$

□

We will see in the next section that for a suitable, stable Hofer geodesic p, $\mathcal{A}_p$ is quasi-flat. However, the quasi-flat condition is much more subtle as we will show.

Remark 2.1.1. *The above sufficient condition for being L^s (locally)-minimizing is likely necessary, but this is left as a question.*

3. Exotic extremals for L^s and Lalonde-McDuff Hamiltonian symplectomorphisms.

We will say that $\{\mathcal{A}_r\}$, $0 \leq r \leq 1$ is a ***strong Cerf homotopy*** of Hamiltonian connections on $M \times S^1$, in degree 2n, with holonomy of $\mathcal{A}_0 = id$, and holonomy of $\mathcal{A}_1 = \widetilde{\phi}$ if the following holds:

- $\{\mathcal{A}_r\}$ is constant for r near 0, 1.
- There are smooth sections u^i of $M \times ([0,1] \times S^1)$ whose restrictions over $r \times S^1 \subset [0,1] \times S^1$ is γ_r^i, where γ_r^i is a flat section of $\mathcal{A}_r$. For some D_0^i, $\widetilde{\gamma}_0^i = (\gamma_0^i, D_0^i)$ is a generator of $CF_{2n}(\mathcal{A}_0)$, while $\widetilde{\gamma}_1^i = (\gamma_1^i, D_1)$ is a generator of $CF_{2n}(\mathcal{A}_1)$, where D_1 is naturally induced by D_0. Moreover we ask that for some collection $\{c_i\}$, $c_i \cdot \widetilde{\gamma}_1^i$ represents $PSS([M]) \in FH_{2n}(\mathcal{A}_1)$.
- The second condition also holds for the family $\{\overline{\mathcal{A}}_r\}$, defined analogously to $\overline{\mathcal{A}}$, with associated sections $\{\overline{u}_j\}$.

Proof of Proposition 1.0.1. Under assumptions of the theorem, there is clearly an extension of $\{\mathcal{A}_r\}$ to a morphism $\mathcal{A} \in \mathcal{P}(id, \widetilde{\phi}_+)$, such that the sections u_i are $\mathcal{A}$-flat. We assume without loss of generality that u_i are regular, i.e. the associated CR operator is surjective for every u_i, otherwise perturb $\mathcal{A}$ so that this is satisfied, see [5, Chapter 8]. Since $\int_{u_i} K_{\mathcal{A}} \pi^* \omega_{st} = A_{H_1}(\widetilde{\gamma})$, it follows that $L_j^+(\mathcal{A})$ is at most $\rho(\widetilde{\phi}, [M])$. By an identical argument $L_j^-(\mathcal{A})$ is at most $-\rho(\widetilde{\phi}^{-1}, [M])$. □

3.1. Lalonde-McDuff symplectomorphisms.

Let $H: S^2 \to \mathbb{R}$ generate a single rotation of S^2, ω in time $1 + \epsilon$. Define H' to be a small time dependent perturbation of $h \circ H$, for $h: \mathbb{R} \to \mathbb{R}$, with $h'' > 0$ and sufficiently large so that the linearized time one flow at the maximum/minimum of H' is overtwisted, i.e. has non-constant periodic orbits with period less than or equal to one. The time dependent perturbation is meant to fix the maximum/minimum of $h \circ H$. The set of Hamiltonian symplectomorphisms formed by time one flow of such an H', is essentially the entire set of Lalonde-McDuff symplectomorphisms (they do not ask for a time dependent perturbation). Set $\mathcal{A}_r$ to be the connection with holonomy path generated by $\eta(r) \cdot H'$, with η as in (1.1).

Lemma 3.1.1. *The family $\{\mathcal{A}_r\}$ is a strong Cerf homotopy in degree 2n.*

Proof. For r less then some threshold r', there is a single generator $\widetilde{\gamma}^r_{\max}$ in $CF_2(\mathcal{A}_r)$, corresponding to the the maximizer max of H'. For r just greater than r' the linearized flow at max becomes overtwisted, (and hence $\widetilde{\gamma}^r_{\max}$ now has CZ index 4) and a new non-constant generator of CZ degree 2 appears in Floer complex of $\mathcal{A}_r$, paired with a new generator of CZ index 3 so that at $r = r'_1$, there is bifurcation in the Floer chain complex. If h'' is sufficiently large, there will be similar bifurcations at $\widetilde{\gamma}^r_{\max}$ for $r > r'_1$ creating more generators and more relations but non of this any longer happens for CZ degree 2. All of this is readily deduced simply from count of generators, (and we know them all explicitely) the fact that CZ degree of $\widetilde{\gamma}^r_{\max}$ goes up by 2 each time it gets more overtwisted, and since the Floer homology is known. An identical argument holds for the family $\{\overline{\mathcal{A}}_r\}$. □

Proof of Theorem 1.2.4. This follows by Proposition 1.0.1, and Lemma 3.1.1 above. □

We end the section by noting that even for morphisms $\mathcal{A}_p$, coming from a quasi-autonomous path $p : [0, 1] \to \text{Ham}(M, \omega)$, which are local minima of the Hofer length functional, $L^s(\mathcal{A}_p)$ may not equal $L_H(p)$.

Example 3.1.2. *The following very simple example was essentially suggested to me by Leonid Polterovich. Let $H : S^2, \omega_{st} \to \mathbb{R}$, be a normalized Hamiltonian generating Hamiltonian flow for a double rotation of S^2, ω_{st} in time $1 + \epsilon$, for a small ϵ, and where $\int_{S^2} \omega_{st} = 1$. Set H' to be the result of a small time dependent perturbation, vanishing at the maximazer* max *of $h \circ H$, where $h'' \leq 0$, s.t the flow at the poles becomes slow, i.e. so that linearized flow at the poles has no non-constant periodic orbits with period less than or equal to 1. Then by Ustilovsky* [6] *time 1 flow of H' generates a stable Hofer geodesic, with L^+ length close to 1. Set $\widetilde{\phi}$ to be the time one map of H', then $\rho(\widetilde{\phi}, [M])$, is at most 1/2. Consequently $PSS([M])$ is not represented by the constant orbit at* max*. It must then be represented by a new non-constant generator $\widetilde{\gamma} \in CF_2(\widetilde{\phi})$. Now for $u \in \overline{\mathcal{M}}(\widetilde{\gamma})$ we have*

$$\int_u K_A \pi^* \omega_{st} = \int_u \eta'(r) \cdot H_p \, \pi^* \omega_{st} =$$
$$\int_0^1 \int_{-\infty}^{\infty} \eta'(r) H(u(r, \theta), \theta) dr \wedge d\theta,$$

this integral is less than $L^+(p)$, unless u is a broken holomorphic section, with principal component the constant section $u_{\max}(z) = \max$, the maximizer of H'. But since we are allowing time dependent perturbations, and H' is assumed to be suitably generic, the 0 dimensional space $\overline{\mathcal{M}}(\widetilde{\gamma})$ will not have any broken elements of this kind. It follows that $L^+_j(\mathcal{A}_p) \neq L^+_H(p)$.

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