ON THE TOPOLOGY OF THE SPACES OF STABLE CURVES

YASHA SAVELYEV

[1]

Abstract. We study here some aspects of the topology of the space of smooth, stable, genus 0 curves in a Riemannian manifold X, i.e. the Kontsevich stable curves, which are not necessarily holomorphic. We use the Hofer-Wysocki-Zehnder polyfold structure on this space and some natural characteristic classes, to show that for $X = BU$, (which is Riemannian in a limiting sense) the rational homology of the spherical mapping space injects into the rational homology of the space of stable curves. We also give here a definition of what we call q-complete symplectic manifolds, which roughly speaking means Gromov-Witten theory captures all information about homology of the space of smooth stable maps.

1. Introduction

The space of smooth stable curves (unparametrized stable maps) of a Riemann surface into a Riemannian manifold appears naturally in the context Gromov-Witten theory in symplectic geometry, particularly in the context of the beautiful polyfold approach of Hofer-Wysocki-Zehnder [8].

The topology of the configuration space of stable curves in a general Riemannian manifold seems very interesting on its own merit. For example we show that the space of based stable curves has the structure of an H-space. This is interesting as the space of unparametrized based spheres in X does not have an H-space structure.

Moreover, this configuration space may also be very natural in the study of gradient flow for the energy functional on the space of smooth maps of say a Riemann sphere into a Riemannian manifold X. It was first observed by Sacks-Uhlenbeck [13] that the flow lines of the resulting parabolic flow often do not converge to smooth maps but rather the associated maps develop bubbling phenomena. This of course presents problems for Morse theory considerations. For example, Eells and Wood [2] show that in a simply connected Kahler manifold X, the only critical points of the energy functional on the mapping space of a Riemann sphere are (anti)-holomorphic maps, which are also absolutely energy minimizing. If Morse theory worked as expected we could conclude that the topology of the mapping space coincides with the topology of the space of absolute minima, i.e. with the space of (anti)-holomorphic maps, which is usually the wrong conclusion. However in a series of remarkable papers [1], [5], [4], [9], [3] (sorry for incomplete list) it is shown that the conclusion becomes essentially correct (for some Kahler manifolds) after a suitable process of stabilization. One possibility for understanding this phenomenon is to partially compactify the space by adding all appropriate stable maps, in such a way that bubbling becomes built in. One may then hope that the gradient flow on the enlarged space satisfies some version of Palais-Smale condition, after

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appropriate completion. It would be most exciting to see if the polyfold theory helps in this.

Our main result concerns injectivity of the map from rational homology of the spherical mapping space into the rational homology of the space of stable curves in the case of $X = BU(r)$. This at least tells us the space of stable curves in that case is topologically non-trivial. Although we are far from understanding the topology of this space.

Finally, we define a notion of q-complete symplectic manifolds, inspired by work Cohen-Segal-Jones [1] studying the spherical mapping space into an almost Kahler manifold by means of certain stabilization of holomorphic spherical mapping spaces.

1.1. **The space of smooth stable maps.** Let $\mathcal{P}^A_{0,n}\,X$ denote the groupoid whose elements are continuous maps into a Riemannian manifold X, g, with total homology class A, from a nodal connected Riemann surface Σ with genus 0, with complex structure j and n marked points. These maps are assumed smooth on each component of Σ, and satisfy some additional conditions:

- If the restriction u_α of u to a component α is non-constant then u_α is not null-homologous.
- If u_α is constant then the number of special points is at least 3.

We will distinguish one marked point by z_0, the other marked points are *unlabeled*. A morphism between a pair of stable maps

$$u_1 : (\Sigma_1, j_1, {z_0}^1) \to X, u_2 : (\Sigma_2, j_2, z_0^2) \to X$$

is a continuous map $\phi : \Sigma_1 \to \Sigma_2$, which is a diffeomorphism on each component, satisfies $\phi^* j_2 = j_1$, maps marked points to marked points, the distinguished marked point z_0^1 to z_0^2 and satisfies $u_2(\phi(z)) = u_1(z)$, for a marked point z. We will call such a pair of stable maps *equivalent*, and equivalence classes of maps the *orbit space*. The orbit space Z, can be endowed with a kind of Deligne-Mumford topology [8, Section 3.2].

We will denote by $\overline{\Omega}^A_{n,\infty}\,X$ the sub-groupoid of smooth stable maps of the polyfold groupoid of stable maps of appropriate regularity, (we will clarify this in Section 2). This groupoid refines $\mathcal{P}^A_{0,n}X$, which means it comes with a natural functor $\overline{\Omega}^A_{n,\infty}\,X \to \mathcal{P}^A_{0,n}X$, inducing homeomorphism on orbit spaces. Note that since we only label one of the marked points, these are slightly different categories than the one considered in [8].

We will refer to elements of $\overline{\Omega}^A_{n,\infty}\,X$ as *smooth stable curves*. We may drop the superscripts A, this will just mean that we are considering all components.

Set

$$\overline{\Omega}_\infty\,X = colim\,\overline{\Omega}_{n,\infty}\,X,$$

the homotopy colimit with the maps in the directed system

$$i_n : \overline{\Omega}_{n,\infty}\,X \to \overline{\Omega}_{n+1,\infty}\,X \tag{1.1}$$

defined as follows. For $u \in \mathcal{P}^A_{0,n}\,X$, $u : (\Sigma, j, \{z_i\}) \to X$, $i_n(u)$ is the map u' from

$$\Sigma' = \Sigma \sqcup (\mathbb{CP}^1, 0, z_1', z_2')/z_0 \sim z_1'.$$

The map $i_n(u)$ is u on the component Σ and constant on the new component. The new distinguished marked point z_0' on Σ' is the marked point 0.

1.1.1. *Product operation.* Let

$$u_1, u_2 : (\Sigma_1, z_0^1), (\Sigma_2, z_0^2) \to X$$

be representatives for a pair of elements $|u_i|$ in $\overline{\Omega}_{n,\infty} X$, respectively $\overline{\Omega}_{m,\infty} X$ such that $u_1(z_0^1) = u_2(z_0^2)$. Then we have a product $|u_1| \star |u_2| \in \overline{\Omega}_{n+m-1} X$ defined as an equivalence class of a map from

$$\Sigma' = \Sigma_1 \sqcup \Sigma_2 \sqcup (\mathbb{CP}^1, 0, z_1', z_2')/z_0^1 \sim z_1', z_0^2 \sim z_2',$$

which is u_1 respectively u_2 on the components Σ_1, respectively Σ_2 and is constant on the new $\mathbb{CP}^1$ component. The new distinguished marked point for Σ' is 0. In other words we concatenate u_1, u_2 with a ghost bubble as intermediary.

This multiplication is homotopy associative, since the main ghost components of domains for $(u_1 \star u_2) \star u_3$, and $u_1 \star (u_2 \star u_3)$, where concatenation takes place correspond to a pair of points in $\overline{M}_{0,4}$, which we may connect by a path. In other words it is conceptually the same argument as the argument for associativity of quantum multiplications.

Moreover the pair of maps:

$$\overline{\Omega}_{n,\infty} X \times \overline{\Omega}_{m,\infty} X \xrightarrow{\star} \overline{\Omega}_{n+m-1,\infty} X \xrightarrow{i_{n+m} \circ i_{n+m-1}} \overline{\Omega}_{n+m+1,\infty} X$$

$$\overline{\Omega}_{n,\infty} X \times \overline{\Omega}_{m,\infty} X \xrightarrow{i_n \times i_m} \overline{\Omega}_{n+1,\infty} X \times \overline{\Omega}_{m+1,\infty} X \xrightarrow{\star} \overline{\Omega}_{n+m+1,\infty} X,$$

coming from two ways of commuting multiplication with stabilization, are homotopy equivalent by a similar argument. Consequently there is an induced map in the homotopy category $\overline{\Omega}_{\infty} X \times \overline{\Omega}_{\infty} X \to \overline{\Omega}_{\infty} X$. (This is a map in the homotopy category of topological groupoids.)

If we ask that our maps u are based, i.e. map the distinguished marked point z_0 to $x_0 \in X$, then the corresponding space $\overline{\Omega}_{x_0,\infty} X$ is a homotopy associative H-space. Consequently homology of $\overline{\Omega}_{x_0,\infty} X$ is a ring with Pontryagin product.

Notation 1.1. *From now on we will be in the above based situation and the subscript x_0 in $\overline{\Omega}_{x_0,\infty}$ will be dropped.*

Note that we have a natural map $\left(\Omega^2 X = Hom_{Top}(\mathbb{CP}^1, X)\right) \to \overline{\Omega}_{3,\infty} X$, by adding three marked points to domain $\mathbb{CP}^1$. And so we have an induced map $\Omega^2 X \to \overline{\Omega}_{\infty} X$. Our main observation in this paper is this:

Theorem 1.2. *The natural map $H_*(\Omega^2 BSU(r), \mathbb{Q}) \to Cob_*(\overline{\Omega}_{\infty} BSU(r), \mathbb{Q})$, is injective for $* \leq 2r - 2$.*

On the right hand side we have bordism groups of a topological groupoid, which are to be defined. In a sense the proof of this theorem concerns certain Gromov-Witten theory of $\mathbb{CP}^r$, and more specifically extension of certain cohomological field theory associated to a fibration $\mathbb{CP}^r \hookrightarrow P \to X$. Usually non virtual moduli cycle methods of Gromov-Witten theory are very successful for such a monotone symplectic manifold like $\mathbb{CP}^r$, however in our geometric setup orbifold language is a must, and consequently dealing with equivariant perturbations is necessary, which in itself requires some kind of virtual moduli cycle. Moreover it is difficult to even formulate our specific Fredholm problem in terms of classical theory, which will hopefully be apparant to the reader, and the abstract Fredholm theory of Hofer-Wysocki-Zehnder, becomes very convenient. Having said all this, we should emphasize that fundamentally we don't see any reason for other approaches to virtual moduli cycle to not be amenable to our needs, although much work and reinventing of wheel might be required. The polyfold language just fits our geometric setup very nicely.

1.2. **Complete symplectic manifolds.** Suppose now (X, ω) is a symplectic manifold. Let $a_i : D_i \to X$, $a_\xi : D_\xi \to \overline{M}_{0,n}$ be smooth maps of closed oriented smooth manifolds, with $\overline{M}_{0,n}$ denoting the moduli space of stable genus 0 Riemann surfaces with n marked points.

Under suitable conditions, for example if (X, ω) is semi-positive we have natural cycles $gw : \overline{\mathcal{M}}_n(A, \{a_i\}, a_\xi) \to \overline{\Omega}\, X$, defined as follows. Consider the diagram below:

$$\begin{array}{ccc} & & (\prod_i D_i) \times D_\xi \\ & & \downarrow{\scriptstyle prod} \\ \overline{\mathcal{M}}_{0,n}^A(X, J) & \longrightarrow & X^n \times \overline{M}_{0,n}, \end{array} \tag{1.2}$$

with $\overline{\mathcal{M}}_{0,n}^A(X, J)$ denoting the compactified moduli space of genus zero, class A, J-holomorphic curves in X, for a regular ω-tamed J. After perturbing the maps to be transverse, we define $\overline{\mathcal{M}}_n^A(X, \{a_i\}, \xi)$ as the pull-back of this diagram (oriented fibre product) and the cycle gw is defined to be the composition of the projection of $\overline{\mathcal{M}}_n^A(X, \{a_i\}, \xi)$ to $\overline{\mathcal{M}}_{0,n}^A(X; J)$, with the tautological map

$$\overline{\mathcal{M}}_{0,n}^A(X, J) \to \overline{\Omega}\, X.$$

For a completely general symplectic manifold (M, ω) the homology class of the cycle gw is represented by a compact branched sub-orbifold, obtained (for example) using abstract Fredholm theory of Hofer-Wysocki-Zehnder. We shall call cycles gw: ***Gromov-Witten*** cycles. To simplify the definition below, we will also call all 0-dimensional cycles into $\overline{\Omega}\, X$ Gromov-Witten cycles.

One of the motivations we had for undertaking study of the space of stable curves, is so we could make the following definition, we say more in the remark below.

Definition 1.3. *We will say that a symplectic manifold (X, ω) is* **q-complete** *if homology of $\overline{\Omega}\, X$ is multiplicatively generated over $\mathbb{Q}$ by homology classes of Gromov-Witten cycles.*

Remark 1.4. *This is the homological version of homotopy approximation of $\Omega^2 X$ by holomorphic mapping spaces from $\mathbb{CP}^1$, which was studied for example in [1], [5], [4], [9], [3]. Given a symplectic manifold X, ω, the basic question is when does*

$$\Omega^2 X \simeq Hol^+(\mathbb{CP}^1, X, \omega, j_X)), \tag{1.3}$$

where Hol^+ denotes the group completion of the C_2-operad space of based j_X-holomorphic maps of $\mathbb{CP}^1$ into X, under the gluing operation. (This is a rough statement.) Remarkably, this is known to be the case for example, for complex projective spaces, generalized flag manifolds and toric manifolds. Unfortunately (1.3) only makes sense for a fixed complex structure j_X i.e. it is a priori not a symplectic property. We wanted a purely symplectic notion, and q-complete symplectic manifolds is one possibility.

Question 1.5. *Is $\mathbb{CP}^n, \omega_{st}$ q-complete?*

This appears to be at the moment a difficult and interesting question. Some interesting and possibly related work is done by Miller [12].

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2. Main argument

Let $X = Gr_{\mathbb{C}}(r, \mathbb{C}^j)$ denote the complex Grassmanians. The topological groupoids $\overline{\Omega}_{n,\infty} X$ of the introduction are topologized as sub-groupoids of the polyfold groupoid $\overline{\Omega}_n X$ of class $(3, \delta_0)$ stable maps with n marked points into X for $0 < \delta_0 < 2\pi$, see [8, Definition 1.1, Section 3]

From now on $\overline{\Omega}_{n,\infty} X$ will denote $\overline{\Omega}^{A=0}_{n,\infty} X$. We are first going to construct using Gromov-Witten theory, degree $2k$ cohomology classes $qc_{k,n,j}$ on the topological groupoids $\overline{\Omega}_{n,\infty} X$ for all n, $2k \leq 2r-2$, $j >> r$, with meaning of "cohomology class" to be explained further below. These will satisfy $qc_{k,n,j} = i_n^* qc_{k,n+1,j}$ and so pass to the limit $\overline{\Omega} X$. Theorem 1.2 will follow once we relate these cohomology classes to Chern classes on $\Omega^2 BSU \simeq BU$.

2.1. Bordism groups of an M-polyfold groupoid.

For a polyfold groupoid G we define its smooth oriented bordism groups as follows.

The group $Cob_k(G)$, is the group of equivalence classes of pairs $(f, \mathcal{X}^k)$, where $f : \mathcal{X}^k \to G_\infty \subset G$, is an sc-smooth functor, with $\mathcal{X}^k$ a dimension k,ètale, proper, stable, smooth, groupoid, also known as orbifold groupoid, see for example [10]. We also assume that $\mathcal{X}^k$ is oriented, compact, without boundary. We say that $g : \mathcal{Y}^k \to G_\infty$ *refines* $f : \mathcal{X}^k \to G_\infty$ if there is an ètale smooth functor $r : \mathcal{Y} \to \mathcal{X}$, inducing a homeomorphism of orbit spaces $|\mathcal{Y}| \to |\mathcal{X}|$, with functors $r \circ f$ and g naturally equivalent. (Related by a natural isomorphism.)

The composition in $Cob_k(G)$ is given by disjoint union. And the equivalence relation is $(f_1, \mathcal{X}_1) \sim (f_2, \mathcal{X}_2)$ if there is an sc-smooth functor of an oriented orbifold groupoid with boundary:

$$F : \mathcal{B}^{k+1} \to G_\infty$$

with $(\partial F, \partial B)$ refining $(f_1, \mathcal{X}_1^{op}) \sqcup (f_2, \mathcal{X}_2)$, where $\mathcal{X}_1^{op}$ denotes $\mathcal{X}_1$ with the opposite orientation.

2.2. Construction of the classes $qc_{k,n,j}$.

Let $\pi : E \to X$ denote the projectivization of the tautological $\mathbb{C}^r$-bundle.

Now given $(f, \mathcal{X})$ representing a class in $Cob_{2k}(\overline{\Omega}_n X)$, let us denote by $\mathcal{E}$ the polyfold groupoid, whose objects are pairs (σ, x), $x \in \mathcal{X}$, and σ a stable section of $E_x = f(x)^* E$, where Σ_x is the domain for the stable map $f(x) : \Sigma_x \to X$. We denote by $\mathcal{E}_x$ the fiber of $\mathcal{E}$ over x. By *stable section* we mean the following. It is a class $(3, \delta_0)$ stable map $\sigma : \Sigma' \to E_x$, of an unmarked nodal genus 0 Riemann surface, with some smooth components of Σ' labeled as *principal*, and some components labeled as *vertical*. The principal components of Σ' are identified with components of Σ_x and the restriction of σ to a principal component, composed with projection π has degree one. The restrictions of σ to vertical components are stable maps, with null-homotopic projection to Σ_x. Since there are no marked points on Σ' the above conditions imply that all vertical components are not null-homotopic as maps into $\mathcal{E}_x$. We will also write $\mathcal{E}_\mathcal{X}$ when we want to emphasize domain of f.

The polyfold structure on $\mathcal{E}$ is a natural adaptation of the main construction in [8]. In fact consider the tautological map $f : \overline{M}_{0,n} \to \overline{\Omega}\, M$, (for (M, ω) some symplectic manifold) the tautological map, defined by taking a representative Σ for $|\Sigma| \in \overline{M}_{0,n}$ and sending it to the constant map to x_0. The polyfold groupoid $\mathcal{E}_{\overline{M}_{0,n}}$ is a slight enlargement of the polyfold of n marked stable maps in M. (These polyfolds would be exactly the same if we made a more strict requirement on vertical components of stable sections to have constant projection to the base, and for principal components to actually be sections. Also in our definition of marked stable maps only z_0 is labeled, so we should really use a slightly different notation for the corresponding moduli space of marked Riemann spheres.)

We now construct a strong polyfold Banach bundle over $\mathcal{E}$. Fix a unitary (in other words $PU(r)$) connection $\mathcal{A}$ on $E \to X$. For $x \in \mathcal{X}$ the restriction $\mathcal{A}_x$ of $\mathcal{A}$ to $E_x = f(x)^* E$ induces almost complex structures $\{J_x\}$ on $\{E_x\}$ in the following standard way. (By almost complex structure on a nodal fibration E_x, we just mean a an almost complex structure on each smooth component, similarly with symplectic forms further on.)

- The natural map $\pi : (E_x, J_x) \to (\Sigma_x, j_x)$ is holomorphic.
- J_x preserves the horizontal sub-bundle of TE_x induced by $\mathcal{A}_x$.
- J_x preserves the vertical tangent bundle $T^{vert}E_x$ of $\mathbb{CP}^{r-1} \hookrightarrow E_x \to \Sigma_x$, and restricts to the standard complex structure on the fibers $\mathbb{CP}^{r-1}$. (That is to say the fibers are identified with $\mathbb{CP}^{r-1}$ up to action of $PU(r)$, which preserves this complex structure.)

The complex structures $\{J_x\}$ are compatible, with symplectic forms $\{\Omega_x\}$ on $\{E_x\}$, constructed for example using the coupling forms of connections $\mathcal{A}_x$, see [6]. There is no real reason for choosing this particular family of almost complex, other then for the sake of having an explicit family. For an element $\sigma \in \mathcal{E}_x$, $\sigma : \Sigma' \to E_x$, the fiber over σ consists of the (appropriate completion of) space of continuous, and smooth over smooth components J_x anti complex linear 1-forms on Σ'_x with values in $\sigma^* TE_x$. By [8, Section 1.2] over the whole $\mathcal{E}$ this can be given the structure of a strong polyfold Banach bundle $\mathcal{W}$. And we have a Fredholm sc-section of $\mathcal{W}$: the Cauchy-Riemann section, by taking the J_x anti-linear part of the differential of $\sigma = (\Sigma' \to E_x)$ as a map into $\sigma^* TE_x$. By classical arguments this section is proper, (has compact 0 locus). We finally define our cohomology classes as certain linear functionals

$$qc_{k,n,j} \in Hom(Cob_{2k}(\overline{\Omega}_n\, X), QH(\mathbb{CP}^{r-1})).$$

We need to say how to compute

$$\langle qc_{k,n,j}, [f] \rangle. \tag{2.1}$$

Recall that each Σ_x comes with a distinguished marked point z_0, which is mapped to a fixed base point in X, (recall definition of $\overline{\Omega}_\infty\, X$). Consequently, we have a smooth family of embeddings $I_x : \mathbb{CP}^{r-1} \to E_x$, which takes $\mathbb{CP}^{r-1}$ to the fiber of E_x over z_0. Let $\overline{\mathcal{M}}([\mathbb{CP}^l], d, \mathcal{E}_{\mathcal{X}})$ denote the *compact* sub-groupoid of $\mathcal{E}_{\mathcal{X}}$ consisting of pairs (σ, x), $x \in \mathcal{X}$ with $\sigma \in \mathcal{E}_{\mathcal{X}}$ in the 0-set of the above constructed Cauchy-Riemann section, whose total degree is d, and which intersects $I_x(\mathbb{CP}^l)$. Where d here is defined as

$$\langle c_1(T^{vert}E_x), \sigma \rangle \cdot \frac{1}{r}. \tag{2.2}$$

Ideally we should also specify f in the notation for the above moduli space, but we omit this where there is no possibility for confusion.

Lemma 2.1. *The expression* (2.2) *is an integer.*

Proof. Let $\{\Sigma_{x,\alpha}\}$ denote smooth components of Σ_x. By assumption that f maps to $A = 0$ component of $\overline{\Omega}^A_{n,\infty} X$, the bundle $\overline{E}_x \to \overline{\Sigma}_x$ formed by taking connected sum of the bundles

$$E_x|_{\Sigma_{x,\alpha}}$$

is

$$\overline{E}_x \simeq \mathbb{CP}^{r-1} \times \overline{\Sigma}_x,$$

as a topological bundle. But then the vertical Chern number of the corresponding glued section σ is r times the degree of the projection of the section to $\mathbb{CP}^{r-1}$, and this of course the same as (2.2). □

Let $\overline{\mathcal{M}}^{reg}([\mathbb{CP}^l], d, \mathcal{E}_{\mathcal{X}})$ denote the *compact* branched sub-orbifold groupoid of $\mathcal{E}_{\mathcal{X}}$, obtained as the zero locus of perturbed CR section by an sc_+ multi-valued perturbation.

The dimension of $\overline{\mathcal{M}}^{reg}([\mathbb{CP}^l], d, \mathcal{E}_{\mathcal{X}})$, is $2dr+2k+2l$. Note that if $0 < 2k \leq 2r-2$, then

$$2dr + 2k + 2l < 0 \text{ unless } d \geq -1.$$

On the other hand $d > 0$ results in too high virtual dimension, and $d = 0$ only contributes to degree 0 class as we will see below. Under this condition:

$$\langle qc_{k,n,j}, [f] \rangle = b \in QH(\mathbb{CP}^{r-1}),$$

where $b \in H_*(\mathbb{CP}^{r-1}, \mathbb{Q})$ is defined by duality:

$$b \cdot [\mathbb{CP}^l] = \# \overline{\mathcal{M}}([\mathbb{CP}^l], -1, \mathcal{E}_{\mathcal{X}}) \in \mathbb{Q},$$

where the left side is the usual intersection product and the right side is the orbifold Gromov-Witten invariant

$$\int_{\overline{\mathcal{M}}^{reg}([\mathbb{CP}^l], -1, \mathcal{E}_{\mathcal{X}})} 1,$$

with $\overline{\mathcal{M}}^{reg}([\mathbb{CP}^l], -1, \mathcal{E}_{\mathcal{X}})$, denoting the regularized smooth compact branched sub-orbifold of $\mathcal{E}_{\mathcal{X}}$ obtained by perturbing the Cauchy-Riemann section, by a class of abstract perturbations called sc_+, see [7, Definition 4.23] This invariant is zero unless the expected dimension is zero.

Lemma 2.2. *The above definition of $qc_{k,n,j}$ is independent of the choice of the representative of f.*

Proof. Let $F : \mathcal{B} \to \overline{\Omega}_n X$ be an sc-smooth functor giving an equivalence between $(f_1, \mathcal{X}_1)$ and $(f_2, \mathcal{X}_2)$. We then have an induced bordism $\overline{\mathcal{M}}^{reg}([\mathbb{CP}^l], -1, \mathcal{E}_{\mathcal{B}})$, of branched sub-orbifolds $\overline{\mathcal{M}}^{reg}([\mathbb{CP}^l], -1, \mathcal{E}_{\mathcal{X}_1})$, $\overline{\mathcal{M}}^{reg}([\mathbb{CP}^l], -1, \mathcal{E}_{\mathcal{X}_2})$.

By Stokes theorem [7, Theorem 4.26],

$$\int_{\overline{\mathcal{M}}^{reg}([\mathbb{CP}^l], -1, \mathcal{E}_{\partial \mathcal{B}})} 1 = 0,$$

On the other hand since $(\partial F, \partial \mathcal{B})$ refines $(f_1^{op}, \mathcal{X}_1^{op}) \sqcup (f_2, \mathcal{X}_2)$, $\mathcal{E}_{\partial \mathcal{B}}$, refines $\mathcal{E}_{\mathcal{X}_1^{op} \sqcup \mathcal{X}_2}$, and $\overline{\mathcal{M}}([\mathbb{CP}^l], -1, \mathcal{E}_{\partial \mathcal{B}})$ refines $\overline{\mathcal{M}}([\mathbb{CP}^l], -1, \mathcal{E}_{\mathcal{X}_1^{op} \sqcup \mathcal{X}_2})$. It follows that

$$\int_{\overline{\mathcal{M}}^{reg}([\mathbb{CP}^l], -1, \mathcal{E}_{\mathcal{X}_1^{op} \sqcup \mathcal{X}_2})} 1 = \int_{\overline{\mathcal{M}}^{reg}([\mathbb{CP}^l], -1, \mathcal{E}_{\partial \mathcal{B}})} 1 = 0.$$

□

2.2.1. *On the use of polyfold theory.* Let us pause here to discuss a bit our use of Polyfold theory in the construction. In a sense it is used in two ways, the first is soft: we need a good topology and structure on $\overline{\Omega}\, X$ to even define what our quantum classes are qualitatively, but this does not use the new abstract Fredholm theory. This point can be avoided for some examples. Say if we only look at orbifold bordism groups of the image of $i_\infty(\Omega^2_A X) \subset \overline{\Omega}\, X$, with $A \neq 0$ denoting homology class and i_∞ the universal map. In this case the definition of $\langle qc_k, [f] \rangle$ is in principle as a straightforward parametric Gromov-Witten invariant, but it is still an orbifold GW invariant, and consequently since we must work equivariantly some kind virtual moduli cycle technique would usually still be needed. (In our case we are just using polyfold techniques.)

From a different angle if we only consider classes in $Cob_k(\overline{\Omega}\, X)$ represented by $f : \mathcal{X} \to \overline{\Omega}\, X$, such that $\mathcal{X}$ is a groupoid with only identity morphisms, then in principle $\langle qc_k, [f] \rangle$ can be defined with classical arguments of McDuff-Salomon, [11], (but a priori it would only be well defined under similarly restricted bordism equivalence relation). However these arguments would have to be adapted to allow dealing with families of fibrations over Riemann surfaces where some base surfaces can become nodal. Moreover we would still have to discuss some polyfold style topology on $\overline{\Omega}\, X$ to describe the qualitative structure of constructed "invariants". To extend all this to the case of general orbifold cycles should in principle be possible, using more classical virtual module cycle techniques but would require much reinvention of the wheel. (For our particular setup.) Polyfold theory gives a more finely tuned approach to our particular geometric situation.

Lemma 2.3. *$qc_{k,n,j} = i_n^* qc_{k,n+1,j}$, and in particular we have induced classes $qc_{k,j} \in Hom(Cob_{2k}(\overline{\Omega}\, X), QH(\mathbb{CP}^{r-1}))$.*

Proof. Let $[f] \in Cob_{2k}(\overline{\Omega}_n\, X)$, $f : \mathcal{X} \to \overline{\Omega}_n\, X$. For the map

$$f' = i_n \circ f : \mathcal{X} \to \overline{\Omega}_{n+1}\, X,$$

each $E'_x = f'(x)^* E$ is $E_x = f(x)^* E$ together with a new smooth component E_{ghost} canonically biholomorphic to $\mathbb{CP}^{r-1} \times \mathbb{CP}^1$, corresponding to the new ghost component of f'. Now by considerations following Lemma 2.1 only total degree $d = -1$ stable holomorphic sections σ can contribute to $\langle qc_{k,n+1,j}, [f'] \rangle$. It follows that we can separate the moduli space $\overline{\mathcal{M}}([\mathbb{CP}^l], -1, \mathcal{E}_{\mathcal{X}}, f')$, into components corresponding to bi-degree (d_f, d_{ghost}), $d_f + d_{ghost} = -1$ induced by the splittings of $\{E'_x\}$ described above. Now $d_f < -1$ components have to contribute 0 to the invariant, since the expected dimension of the corresponding (component of) moduli space would be negative by considerations above. On the other hand $d_{ghost} < 0$ simply give an empty component since $\mathbb{CP}^{r-1} \times \mathbb{CP}^1$ has no stable holomorphic sections with negative degree. Consequently the only possibility is $d_f = -1, d_{ghost} = 0$, but then the conclusion obviously follows. □

Proof of Theorem 1.2. By Milnor-Moore, Cartan-Serre theorems the rational homology of $\Omega^2 BSU(r) \simeq \Omega SU(r)$ is the free graded commutative Pontryagin algebra generated by spherical classes. Given $f : S^{2k} \to \Omega^2 BSU(r)$, we can factor it through a map to $\Omega^2 Gr_{\mathbb{C}}(r, \mathbb{C}^j)$ for j large enough. The pull-back of the classes $qc_{k,j}$ on $\overline{\Omega}\, Gr_{\mathbb{C}}(r, \mathbb{C}^j)$ by the natural map $\Omega^2 Gr_{\mathbb{C}}(r, \mathbb{C}^j) \to \overline{\Omega}\, Gr_{\mathbb{C}}(r, \mathbb{C}^j)$ make sense as linear functionals $Hom(H_*(\Omega^2 Gr_{\mathbb{C}}(r, \mathbb{C}^j), \mathbb{Q}), QH(\mathbb{CP}^{r-1}))$ by above characterization of the rational homology. The induced classes in $H^*(\Omega^2 BSU(r), QH(\mathbb{CP}^{r-1}))$ are by construction the stable quantum classes considered in [14], except of course we did not need orbifold cycles but worked with cycles that are maps of closed oriented manifolds. We paraphraze the main theorem of [14] as follows:

Theorem 2.4 ([14])**.** *If $2k \leq 2r - 2$, then $0 \neq a \in H_{2k}(\Omega^2 BSU(r), \mathbb{Q})$ if and only if for some $\{\beta_i, \alpha_i\}$*

$$0 \neq \langle \prod_i qc_{\beta_i}^{\alpha_i}, a \rangle, \ \text{where} \sum_i 2\beta_i \cdot \alpha_i = 2k.$$

Consequently the theorem follows immediately by Theorem 2.4. □

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Centre de Recherches Mathématiques, Université de Montréal, C.P. 6128, Succ. Centre-ville, Montréal H3C 3J7, Québec, Canada

E-mail address: savelyev@crm.umontreal.ca