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VIRTUAL MORSE THEORY ON $\Omega\mathbf{Ham}(M,\omega)$

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ABSTRACT. We relate previously defined quantum characteristic classes to Morse theoretic aspects of the Hofer length functional on $\Omega\mathrm{Ham}(M,\omega)$. As an application we prove a theorem which can be interpreted as stating that this functional behaves "virtually" as a perfect Morse-Bott functional with a flow. This can be applied to study topology and Hofer geometry of $\mathrm{Ham}(M,\omega)$. We also use this to give a prediction for the index of some geodesics for this functional, which was recently partially verified by Yael Karshon and Jennifer Slimowitz.

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1. Introduction

Hamiltonian fibrations over Riemann surfaces form a rich object from the point of view of Gromov Witten theory, as the properties of holomorphic sections of such fibrations can be closely tied with the underlying geometry of the fibration. Here by geometry we may mean a number of things like curvature properties of Hamiltonian connections, geometry of the coupling forms in the sense of [3], as well as the associated quantities like Gromov's K-area, (cf. [12]) and the related notion of area of a Hamiltonian fibration.

Gromov Witten theory of Hamiltonian fibrations $M \hookrightarrow X \xrightarrow{\pi} \Sigma$ fits into a certain $2d$ Hamiltonian cohomological field theory, or even more generally into a string background [15], considerably extending Gromov Witten theory of (M,ω). In this paper we will be concerned with a fairly small, but geometrically still rich part of this theory by restricting to genus 0, one input one output part of the data of field theory. With some further restrictions, the data one gets is a ring homomorphism defined in [16]:

$$\Psi : H_*(\Omega\mathrm{Ham}(M,\omega),\mathbb{Q}) \to QH(M).$$

Geometry of Hamiltonian fibrations over S^2 can be tied with Hofer geometry of loops in $\mathrm{Ham}(M,\omega)$, and we are going to relate Ψ to a kind of virtual Morse theory of the positive Hofer length functional, $L^+ : \Omega\mathrm{Ham}(M,\omega) \to \mathbb{R}$, (see (2.2)).

1.1. Morse theory on $\Omega\mathbf{Ham}(M,\omega)$ and Ψ.

Let $h : B \to \Omega\mathrm{Ham}(M,\omega)$ be a smooth cycle, where B is a closed oriented smooth manifold. Let

$$P_h = B \times M \times D_0^2 \bigcup B \times M \times D_\infty^2 / \sim, \tag{1.1}$$

where $(b,x,1,\theta)_0 \sim (b,h_{b,\theta}(x),1,\theta)_\infty$, using the polar coordinates $(r,2\pi\theta)$. We get a bundle

$$p : P_h \to B,$$

with fiber modelled by a Hamiltonian fibration $M \hookrightarrow X \xrightarrow{\pi} S^2$. A Hamiltonian connection or equivalently, a coupling form (see [9, Theorem 6.21]) on the Hamiltonian bundle

$$M \hookrightarrow P_h \to B \times S^2 \tag{1.2}$$

induces a family of complex structures $\{X_b = p^{-1}(b), J_b\}$. The map Ψ is defined by counting fiber-wise (i.e. vertical) holomorphic curves in $p : P_h \to B$, with some constraints. The details are given in Section 2.

Let $\gamma : S^1 \to \text{Ham}(M, \omega)$ be a one parameter subgroup, which is always assumed in this paper to be generated by a Morse Hamiltonian H. The loop γ is a smooth point of the functional L^+ and is critical, see Ustilovsky [17]. Our focus will be on smooth cycles $h : B \to \Omega\text{Ham}(M, \omega)$ that resemble unstable manifolds for the functional L^+ of γ, in the sense below.

Definition 1.1. *We say that a smooth map $h : B \to \Omega Ham(M, \omega)$ is Morse at γ, if the pullback of L^+ to B attains its maximum at a unique point* $\max \in B$ *such that L^+ is Morse at* $\max$ *and* $h(\max) = \gamma$.

To make use of this structure we construct a coupling form on P_h naturally adapted to the above properties of h. As a consequence, for the induced family $\{J^h_b\}$ of complex structures on P_h all vertical holomorphic curves in P_h of "maximum allowed c-energy" (Definition 3.3) localize over max and in fact correspond to a single, distinguished flat section $\sigma_{\max}$ of the fiber $X_{\max}$.

One would then like that this curve is persistent and contributes to the invariant Ψ. However, without further restriction on (h, B) this is not true as one can see from simple heuristic intuition: the map $h : B \to \Omega\text{Ham}(M, \omega)$ which by assumption is Morse at γ can be homotoped below the energy level $L^+(\gamma)$, unless its dimension is that of the unstable manifold of γ for the functional L^+, (provided one can make sense of such an unstable manifold). Once this happens, the part of the invariant Ψ corresponding to curves of the same class as $\sigma_{\max}$ has to vanish. (This is explained in the proof of Theorem 1.5). There is a more formal obstruction: dimension of B has to be the dimension of the cokernel of the linearized Cauchy-Riemann operator corresponding to the pair $(\sigma_{\max}, X_{\max})$, for otherwise the index of the overall problem is not zero and our moduli space does not even have the expected dimension.

The index of the above Cauchy-Riemann operator will be denoted by $I^{virt}(\gamma)$ and we call this the *virtual index of* γ. Indeed, the name arises from intuition that the above necessary conditions are the same and this is verified by Theorem 1.5 below. We show in [16, Section 5.1]:

$$I^{virt}(\gamma) = \sum_{\substack{1 \le i \le n \\ k_i \le -1}} 2(|k_i| - 1), \tag{1.3}$$

where k_i are the *weights* of the linearized action of γ on $T_{x_{\max}} M$, the tangent space to the maximum: $x_{\max}$, of the generating function H of γ. This is a single point since H is Morse and the level sets of H must be connected by the Atiyah-Guillemin-Sternberg convexity theorem. To define these weights one takes an S^1 equivariant orientation preserving identification of $T_{x_{\max}}$ with $\mathbb{C}^n$, which splits into γ invariant 1 complex dimensional subspaces N_{k_i}, on which γ is acting by

$$v \mapsto e^{2\pi i k_i \theta} v. \tag{1.4}$$

These k_i are then defined to be the weights of the circle action γ. Our conventions are

$$\omega(X_H, \cdot) = -dH(\cdot) \tag{1.5}$$

$$\omega(\cdot, J\cdot) > 0. \tag{1.6}$$

With these conventions the above weights are negative. Let $L^+(\gamma)$ denote the positive Hofer length of γ, see Section 2.1. The following is our main main technical result proved in in Section 3:

Theorem 1.2. *Let $h : B_\gamma \to \Omega Ham(M,\omega)$ be Morse at γ and such that $I^{virt}(\gamma) = \dim B_\gamma$, then*

$$0 \neq \Psi(h) = [\pm pt] \cdot e^{iL^+(\gamma)} + corrections \quad \in QH(M). \tag{1.7}$$

In [16] we also studied what we called the max length measure of $h : B \to \Omega\mathrm{Ham}(M,\omega)$ which is defined by

$$L^+(h) \equiv \max_{b \in B} L^+(h(b)). \tag{1.8}$$

Corollary 1.3. *Let $h : B_\gamma \to \Omega Ham(M,\omega)$ be as in Theorem 1.2, then the cycle $h : B_\gamma \to \Omega Ham(M,\omega)$ does not vanish in rational homology and moreover minimizes the max length measure in its homology class.*

When γ is generated by a Morse Hamiltonian H, γ is a smooth point of L^+ by Ustilovsky's work [17] and we can make the following definition.

Definition 1.4. *Let γ be a one parameter subgroup of $Ham(M,\omega)$, where (M,ω) is any closed symplectic manifold. We define the Hofer index $I^H(\gamma)$ to be the maximal dimensional subspace of $T_\gamma\Omega Ham(M,\omega)$ on which the Hessian of L^+ is negative definite.*

Of course the above index can apriori be infinite. On the other hand we have:

Theorem 1.5. *Let $h : B_\gamma \to \Omega Ham(M,\omega)$ be as in Theorem 1.2 then*

$$I^H(\gamma) = I^{virt}(\gamma) = \dim B_\gamma.$$

Conjecture 1.6. *Let γ be a Hamiltonian circle action on (M,ω) generated by a Morse Hamiltonian, then*

$$I^H(\gamma) = I^{virt}(\gamma). \tag{1.9}$$

Yael Karshon and Jennifer Slimowitz [5] have recently verified that $I^H(\gamma) \geq I^{virt}(\gamma)$. They explicitly construct a local family deforming γ of dimension $I^{virt}(\gamma)$, so that the Hessian of L^+ on the tangent space to this family is negative definite. Following a suggestion of Leonid Polterovich I know expect to be able to prove this conjecture, using classical calculus of variations and Duistermaat's theorem relating Morse index and Maslov index, [2].

1.2. **Hofer functional and "virtual Morse theory".** Given the conjecture above, Corollary 1.3 can be restated with Morse index of γ replacing virtual index of γ. However, this now raises an observation. Since the Morse indexes are even, the first part of the statement of the corollary would hold automatically if Hofer length functional had well behaved negative gradient flow and an associated Morse-Bott complex computing homology of $\Omega\mathrm{Ham}(M,\omega)$. Nothing like this could remotely be true directly as the Hofer length functional is extremely degenerate and poorly behaved analytically. Yet from the point of view of Corollary 1.3 something like this happens virtually.

1.3. **Generalizations to path spaces.** All of the results outlined in this section have appropriate generalizations to path spaces. For example Theorem 1.2 can be stated for cycles in path space Ω_{ϕ_1,ϕ_2} Morse at γ, where γ is an autonomous geodesic (generated by Morse H) between non-conjugate $\phi_1, \phi_2 \in \mathrm{Ham}(M,\omega)$. Here non-conjugate is in Hofer geometry sense, which amounts to the condition that the linearized flow at the maximizer of H has no periodic orbits with period less than 1. In this case one must use Floer homology instead of quantum homology, and the role of fibrations $M \hookrightarrow X_b \to S^2$ is played by fibrations

$$M \hookrightarrow X_b \to \mathbb{R} \times S^1$$

with asymptotic ($r \mapsto \pm\infty$, with r, θ coordinates on $\mathbb{R} \times S^1$) boundary monodromy maps ϕ_1, ϕ_2, but otherwise the statements and proofs are completely analogous. It is even likely that one can extend from autonomous geodesics to any geodesics of the Hofer length functional, (which are generated by quasi-autonomous time dependent Hamiltonians, see [17] and [8]). However in this case the proofs may need to slightly change, since we give up some symmetry.

1.4. **A special case: $\Omega\mathbf{Ham}(G/T)$.** Consider the Hamiltonian action of G on G/T. In Section 4 we relate the "Morse theory" for the functional $L^+ : \Omega G \to \mathbb{R}$ pulled back from $\Omega\mathrm{Ham}(G/T)$ and the energy functional $E : \Omega G \to \mathbb{R}$ induced by a bi-invariant metric on G. The latter functional is amazingly well behaved. It is Morse-Bott, the "smooth" negative gradient flow (i.e. energy flow) exists for all time and the unstable manifolds of critical level sets are complex submanifolds. (These appear to be rather deep facts of life, see [14] and or [13].)

Let $f : \Omega G \to \Omega\mathrm{Ham}(G/T)$ be the map induced by the Hamiltonian action. The first theorem follows from Corollaries 4.2, 1.3.

Theorem 1.7. *Let G be a semi simple Lie group, γ an S^1 subgroup of G whose centralizer is the torus, and $h : B_\gamma \to G$ the pseudocycle corresponding to the unstable manifold of γ in ΩG for the Riemannian energy functional. Then the pseudocycle $f \circ h : B_\gamma \to \Omega Ham(G/T)$ is non-vanishing in $H_{\dim B_\gamma}(\Omega Ham(G/T), \mathbb{Q})$ and moreover it minimizes the max-length measure in its homology class, (see eq.* (1.8)*).*

Theorem 1.5 together with Corollary 4.2 gives:

Theorem 1.8. *Let γ be as in the above theorem, then $I^H(f \circ \gamma)$ is the Riemannian index of γ, i.e. the index of the geodesic γ as a critical point of the Riemannian energy functional on ΩG.*

Alexander Givental asked me the following natural question:

Question 1.9. *Does the first part of Theorem 1.7 remain true if $\Omega Ham(G/T)$ is replaced with $\Omega\mathrm{Diff}(G/T)$?*

My feeling is that the answer is no, however not much is known about topology of diffeomorphism groups of higher dimensional manifolds.

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2. Preliminaries and the map Ψ

2.1. **The group of Hamiltonian symplectomorphisms and Hofer metric.** Given a smooth function $H_t : M^n \to \mathbb{R}$, $0 \leq t \leq 1$, there is an associated time dependent Hamiltonian vector field X_t, $0 \leq t \leq 1$, defined by

$$\omega(X_t, \cdot) = -dH_t(\cdot). \tag{2.1}$$

The vector field X_t generates a path $\gamma : [0,1] \to \mathrm{Diff}(M,\omega)$. Given such a path γ, its end point $\gamma(1)$ is called a Hamiltonian symplectomorphism. The space of Hamiltonian symplectomorphisms forms a group, denoted by $\mathrm{Ham}(M,\omega)$.

In particular the path γ above lies in $\mathrm{Ham}(M,\omega)$. It is well-known that any smooth path γ in $\mathrm{Ham}(M,\omega)$ with $\gamma(0) = id$ arises in this way (is generated by $H_t : M \to \mathbb{R}$). Given such a] γ, the *Hofer length*, $L(\gamma)$ is defined by

$$L(\gamma) := \int_0^1 \max_M(H_t^\gamma) - \min_M(H_t^\gamma)dt,$$

where H_t^γ is a generating function for the path $\gamma(0)^{-1}\gamma(t)$, $0 \leq t \leq 1$. The Hofer distance $\rho(\phi,\psi)$ is defined by taking the infimum of the Hofer length of paths from ϕ to ψ. It is a deep theorem that the resulting metric is non-degenerate, (cf. [4, 7]). This gives $\mathrm{Ham}(M,\omega)$ the structure of a Finsler manifold. We will be more concerned with a related measure of the path,

$$L^+(\gamma_t) := \int_0^1 \max(H_t^\gamma)dt, \tag{2.2}$$

where H_t^γ is in addition normalized by the condition

$$\int_M H_t^\gamma \omega^n = 0.$$

2.2. **Quantum Homology.** For a monotone symplectic manifold (M,ω) we set $QH(M) = H_*(M,\mathbb{C})$. For us this is an ungraded vector space with a special product called quantum product. For integral generators $a, b \in H_*(M)$, this is the product defined by

$$a * b = \sum_{A \in H_2(M)} b_A e^{-i\omega(A)}, \tag{2.3}$$

where b_A is the homology class of the evaluation pseudocycle from the moduli space of pointed J-holomorphic curves in class A, intersecting generic pseudocycles representing a, b, for a generic ω tamed J. This sum is finite in the monotone case: $\omega = kc_1(TM)$, with $k > 0$. The product is then extended to $QH(M)$ by linearity. For more technical details see [10].

2.3. **Quantum characteristic classes.** Here, we give a brief overview of the construction of the map

$$\Psi : H_*(\Omega\mathrm{Ham}(M,\omega),\mathbb{Q}) \to QH(M),$$

originally defined in [16] and which is a natural generalization of Seidel representation. Let $h : B \to \Omega\mathrm{Ham}(M,\omega)$ be a smooth cycle (the associated map $h : B \times S^1 \to \mathrm{Ham}(M,\omega)$ is smooth), where B is a closed oriented smooth manifold, and let $p : P_h \to B$ be as in equation (1.1).

Fix a family $\{j_{b,z}\}$ of almost complex structures on

$$M \hookrightarrow P_h \to B \times S^2,$$

fiberwise compatible with ω. Given a smooth family $\{\mathcal{A}_b\}$, $\mathcal{A}_b$ is a Hamiltonian connection on $X_b = p^{-1}(p)$ we have an induced family of complex structures $\{J_b^{\mathcal{A}}\}$ defined as follows.

- The natural map $\pi : (X_b, J_b^{\mathcal{A}}) \to (S^2, j)$ is holomorphic for each b.
- $J_b^{\mathcal{A}}$ preserves the horizontal subbundle $Hor_b^{\mathcal{A}}$ of TX_b induced by $\mathcal{A}$.
- $J_b^{\mathcal{A}}$ preserves the vertical tangent bundle of $M \hookrightarrow P_h \to B \times S^2$ and restricts to the family $\{j_{b,z}\}$.

Definition 2.1. *A family $\{J_b\}$ is called* **π-compatible** *if it is $\{J_b^{\mathcal{A}}\}$ for some connection $\mathcal{A}$ as above.*

The importance of this condition is that it forces bubbling to happen in the fibers of $M \hookrightarrow X_b \to S^2$, where it is controlled by monotonicity of M, ω.

Remark 2.2. *For the most part we work with a fixed symplectic manifold (M,ω), which we will assume to be monotone. Also, for the purpose of the following definition the family $\{j_{b,z}\}$ is fixed. However, it will be helpful in Section 4 to vary the families $\{j_{b,z}\}$, $\{\omega_{b,z}\}$ on $M \hookrightarrow P_h \to B \times S^2$, so long as each fiber $(M_{b,z}, \omega_{b,z}, j_{b,z})$ is Fano, i.e. $c_1(TM)$ is positive on $j_{b,z}$ holomorphic curves. This will be done not for any compactness or regularity reasons, but for other geometric reasons. This will vary the notion of a π-compatible family $\{J_b\}$, but not the map Ψ below.*

The map Ψ we now define measures part of the degree of quantum self intersection of a natural submanifold $B \times M \subset P_f$. The entire quantum self intersection is captured by the total quantum class of P_f, discussed in [16]. We define Ψ as follows:

$$\Psi([B,f]) = \sum_{A \in j_*(H_2^{sect}(X))} b_A \cdot e^{-i\mathcal{C}(A)}. \tag{2.4}$$

Here,

- $H_2^{sect}(X)$ denotes the section homology classes of X.
- $\mathcal{C}$ is the coupling class of Hamiltonian fibration $M \hookrightarrow P_f \to B \times S^2$, see [6, Section 3]. Its restriction to the fibers $X \subset P_f$ is uniquely determined by the condition

$$i^*(\mathcal{C}) = [\omega], \qquad \int_M \mathcal{C}^{n+1} = 0 \in H^2(S^2). \tag{2.5}$$

  where $i : M \to X$ is the inclusion of fiber map, and the integral above denotes the integration along the fiber map for the fibration $\pi : X \to S^2$.
- The map $j_* : H_2^{sect}(X) \to H_2(P_f)$ is induced by inclusion of fiber.

- The coefficient $b_A \in H_*(M)$ is defined by duality:

$$b_{\widetilde{A}} \cdot_M c = ev_0 \cdot_{B\times M} [B] \otimes c,$$

where

$$ev_0 : \mathcal{M}_0(P_h, \widetilde{A}, \{J_b\}) \to B \times M$$
$$ev_0(u,b) = (u(0), b)$$

denotes the evaluation map from the space

$$\mathcal{M}(P_f, A, \{J_b\}) \tag{2.6}$$

of pairs (u,b), u is a J_b-holomorphic section of X_b in class A and $\cdot_M, \cdot_{B\times M}$ denote the intersection pairings.

- The family $\{J_b\}$ is π-compatible in the sense above.

3. Hofer geometry and Ψ

We now show how to construct a π-compatible family $\{J_b^h\}$ on P_h naturally adapted to Hofer geometry of the map $h : B \to \Omega\mathrm{Ham}(M,\omega)$. This family is induced from a family of Hamiltonian connections $\{\mathcal{A}_b, X_b\}$, which are in turn induced by a family of certain closed forms $\{\widetilde{\Omega}_b^h\}$, which we now describe.

The construction of this family mirrors the construction in Section 3.2 of [16]. First we define a family of forms $\{\widetilde{\Omega}_b^\infty\}$ on $B \times M \times D^2_\infty$.

$$\widetilde{\Omega}_b^h|_{D^2_\infty}(x,r,\theta) = \omega - d(\eta(r)H_\theta^b(x)) \wedge d\theta \tag{3.1}$$

Here, H_θ^b is the generating Hamiltonian for $h(b)$, normalized so that

$$\int_M H_\theta^b \omega^n = 0,$$

for all θ and the function $\eta : [0,1] \to [0,1]$ is a smooth function satisfying

$$0 \leq \eta'(r),$$

and

$$\eta(r) = \begin{cases} 1 & \text{if } 1-\delta \leq r \leq 1, \\ r^2 & \text{if } r \leq 1-2\delta, \end{cases}$$

for a small $\delta > 0$.

Note that under the gluing relation $\sim$, $(x,1,\theta)_0 \mapsto (h(b,\theta)x, 1, \theta)_\infty$. Thus, $\frac{\partial}{\partial\theta} \mapsto X_{H_\theta^b} + \frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial x} \mapsto (\gamma_\theta)_*(\frac{\partial}{\partial x})$, and moreover $\frac{\partial}{\partial r} \mapsto -\frac{\partial}{\partial r}$. We leave it to the reader to check that the gluing relation $\sim$ pulls back the form $\widetilde{\Omega}_b^h|_{D^2_\infty}$ to the form ω on the boundary $M \times \partial D^2_0$, which we may then extend to ω on the whole of $M \times D^2_0$. Let $\{\widetilde{\Omega}_b^h\}$ denote the resulting family on X_b. The forms $\widetilde{\Omega}_b^h$ on X_b restrict to ω on the fibers M and the 2-form $\int_M (\widetilde{\Omega}_b^h)^{n+1}$ vanishes on S^2. Such forms are called *coupling forms*, which is a notion due to Guillemin Lerman and Sternberg [3]. The form $\widetilde{\Omega}_b^h$ induces a connection on X_b, by declaring horizontal subspaces to be those which are $\widetilde{\Omega}_b^h$-orthogonal to the vertical tangent spaces of $\pi : X_b \to S^2$.

Remark 3.1. *The induced connection is Hamiltonian and moreover every Hamiltonian connection on X_b is induced by a unique coupling form in above sense, see* [9, Theorem 6.21].

We denote by $\{J^h_b\}$ the induced family of complex structures. An important property of the family $\{J^h_b\}$ is that it is almost compatible with the family $\{\Omega^h_b, X_b\}$ defined by

$$\Omega^h_b|_{D^2_\infty} = \widetilde{\Omega}^h_b|_{D^2_\infty} + (\max_x H^b_\theta(x)) d\eta \wedge d\theta, \quad \Omega^h_b|_{D^2_0} = \widetilde{\Omega}^h_b|_{D^2_0}. \tag{3.2}$$

Where, *almost compatible* means that $\Omega^h_b(v, J^h_b) \geq 0$, $v \in TX_b$ and this inequality is strict for $v \in T^{vert}X_b$.

By the characterization of the class $\mathcal{C}$ in (2.5):

$$[\widetilde{\Omega}^h_b] = j^*(\mathcal{C}). \tag{3.3}$$

Thus,

$$[\Omega^h_b] = j^*(\mathcal{C}) + [\pi^*(\alpha_b)], \tag{3.4}$$

where $j : X_b \to P_h$ is the inclusion map, and α_b is an area form on S^2 with

$$L^+(h(b)) = \int_{S^2} \alpha_b. \tag{3.5}$$

Lemma 3.2. *Let $\{\Omega^h_b\}$ and $\{J^h_b\}$ be as above, then we have the property that a vertical J^h_b-holomorphic section u in the fiber $X_b \subset P_h$ gives a lower bound*

$$-\mathcal{C}([u]) \leq L^+(h(b). \tag{3.6}$$

Proof. Let $u : S^2 \to X_b \subset P_h$ be a holomorphic section, then

$$0 \leq \Omega^h_b([u]). \tag{3.7}$$

This follows from the almost compatible condition on Ω^h_b and J^h_b, from the fact that u is a holomorphic map and by the fact that J^h_b is π-compatible. Combining (3.7) with (3.4), (3.5) we get:

$$0 \leq \int_u \Omega^h_b = \mathcal{C}([u]) + L^+(h(b)).$$

□

Definition 3.3. *As the quantity $-\mathcal{C}([u])$ is so important for us, we give it a name:* c-energy *of u, or the* **coupling energy** *of u.*

Note that there are no fiber holomorphic curves in P_h in class A with c-energy$(A) > H_{\max}$, the maximum of the generating function H of γ. This follows from the assumption that the pullback of the functional L^+ to B_γ attains its unique maximum at max, $L^+(h(\max)) = L^+(\gamma) = H_{\max}$ and from the energy inequality (3.6). On the other hand there is a special class $A_{\max} \in H_2(X_\gamma) \subset H_2(P_h)$ for which c-energy$(A_{\max}) = H_{\max}$. We now describe this.

Let $x_{\max}$ denote the unique max of H, (cf. the discussion preceding Theorem 1.2). There is a corresponding $\Omega^h_{\max}$-horizontal and thus holomorphic section $\sigma_{x_{\max}}$ of $\pi : X_{\max} \to S^2$,

$$\sigma_{x_{\max}}(z) = (\{x_{\max}\}, z)_{0,\infty} \subset M \times D^2_{0,\infty} \subset X_{\max} \quad \text{for } z \in D^2_{0,\infty}. \tag{3.8}$$

By (3.1), (3.3) c-energy$([\sigma_{x_{\max}}]) = H_{\max}$. Moreover, there are no other holomorphic sections u in $X_{\max}$ with c-energy$([u]) = H_{\max}$. This observation is due to

Seidel. For suppose otherwise, then by the proof of Lemma 3.2 we must have that $\int_u \Omega^h_{\max} = 0$, and so

$$0 = \int_u \omega - \eta(r)dH \wedge d\theta - \int_u H d\eta \wedge d\theta + (\sup_x H(x)) d\eta \wedge d\theta. \tag{3.9}$$

Note that u is necessarily horizontal, for otherwise $\int_u \Omega^h_{\max} > 0$ (by the almost compatible property of $\Omega^h_{\max}$ and $J^h_{\max}$ and π-compatible property of $J^h_{\max}$). Hence the form $\omega - \eta(r)dH \wedge d\theta$ must vanish on u, as the horizontal subspaces are spanned by vectors $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} + \eta(r)X_H$. Thus, (3.9) can only happen if u is $\sigma_{x_{\max}}$. In particular, the space $\mathcal{M}_0(X_{\max}, J^h_{\max}, A_{\max})$ of unmarked holomorphic sections in $X_{\max}$ in homology class $A_{\max} = [\sigma_{\max}]$ is identified with the point $x_{\max}$. We will also denote by $A_{\max}$ the class $j_*(A_{\max}) \in H_2(P_h)$, where $j : X \to P_h$ is the inclusion of fiber map.

Proposition 3.4. *Let $h : B_\gamma \to \Omega Ham(M,\omega)$ be a smooth oriented cycle and $\{\Omega^h_b\}$ as above. Suppose that the pullback by h of the function L^+ to B_γ attains its maximum at the unique point* $\max \in B$, *such that* $h(\max) = \gamma$ *then $\mathcal{M}_0(P_h, A_{\max}, \{J^h_b\})$ lies over* $\max$ *and is identified with $x_{\max}$.*

Proof. The energy inequality (3.6) shows that any holomorphic curve in P_h with c-energy $H_{\max}$ must lie in the fiber $X_{\max}$. Consequently, by the discussion preceding the proposition,

$$\mathcal{M}_0(P_h, \{J^h_b\}, A_{\max}) \simeq \mathcal{M}_0(X_{\max}, J^h_{\max}, A_{\max}) \simeq x_{\max}.$$

□

3.1. **Proof of Theorem 1.2.** One example of Theorem 1.2 that the reader may keep in mind comes from the Hamiltonian S^3 action on S^2. Take γ to be a one parameter subgroup of S^3, which is a great geodesic going around S^3 once. The subgroup γ then acts on S^2 by rotating it twice. The induced loop $f \circ \gamma \subset \Omega\mathrm{Ham}(S^2)$ is a critical point of L^+, and this loop has a two parameter family of shortenings. More specifically, the unstable manifold B_γ of γ in ΩS^3 (for the Riemannian energy functional) is two dimensional (by the Riemannian index theorem), and the pullback of the positive Hofer length functional to B_γ is Morse at its maximum $\max \subset B_\gamma$. Why all this is the case will be explained in the next section.

Proof. It will be helpful to work with a special subset of π-compatible families $\{J_b\}$. Let $\widetilde{C}_b$ denote the space of coupling forms on X_b, (see Remark 3.1) which restrict to ω over $M \times D^2_0$ in the notation of (1.1). A general element $\widetilde{\Omega} \in \widetilde{C}_b$ is determined by a pair of families of functions $G^{\eta,\theta}, F^{\eta,\theta} : M \to \mathbb{R}$,

$$\widetilde{\Omega}|_{D^2_\infty} = \widetilde{\Omega}^h_b|_{D^2_\infty} + d(G^{\eta,\theta} d\theta) + d(F^{\eta,\theta} d\eta),$$

where $\widetilde{\Omega}^h_b|_{D^2_\infty}$ is defined in (3.1) and η is defined in the discussion following (3.1).

We set $C_b \subset \widetilde{C}_b$ to be the subspace of those forms for which

$$G^{\eta,\theta} = \eta \cdot G^\theta, \text{ for some } G_\theta \text{ and } \frac{d}{d\theta} F^{\eta,\theta} = 0.$$

We also set C to be the space of families $\{\widetilde{\Omega}_b\}$ on P_h, with each $\widetilde{\Omega}_b \in \mathcal{C}_b$. We have a function

$$\mathrm{area} : C_b \to \mathbb{R}, \tag{3.10}$$

$$\text{area}(\widetilde{\Omega}) = \inf\{\int_{S^2} \alpha \,|\widetilde{\Omega} + \pi^*(\alpha) \text{ is symplectic}\}. \tag{3.11}$$

Let $\{\widetilde{\Omega}'_b\}$ be as in Lemma 3.6 and sufficiently C^∞-close to $\{\widetilde{\Omega}^h_b\}$, then $\{\widetilde{\Omega}'_b\}$ has the property that the function

$$b \mapsto \text{area}(\widetilde{\Omega}'_b) \tag{3.12}$$

on B_γ attains its maximum at the unique point max $\in B_\gamma$ and this function is Morse at max. This readily follows from the assumption that the pullback of L^+ to B_γ is Morse at max, and by Lemma 3.5 below. Let $\{J'_b\}$ be the family induced by $\{\widetilde{\Omega}'_b\}$, then by the proof of Proposition 3.4 $\mathcal{M}_0(P_{h'}, A_{\max}, \{J'_b\}) \simeq F_{\max} = \max$ and this moduli space is regular by construction. We thus verified the leading term of $\Psi(h)$ up to sign, which depends on the orientation of the cycle B_γ. The corrections are in lower c-energy classes A, and consequently give rise to higher dimensional moduli spaces via the dimension formula

$$2n + \dim B_\gamma + < 2c_1(T^{vert} X), A >, \tag{3.13}$$

and therefore are linearly independent of the leading term, (if they contribute). $\square$

Lemma 3.5. *The coupling form $\widetilde{\Omega}^h_{\max}$ is a smooth point of the* area *functional on $C_{\max}$ and is critical.*

Variation $\widetilde{\Omega}^s$ in $C_{\max}$ induces a variation of the boundary monodromy maps $\gamma_s : [0,1] \to \text{Ham}(M,\omega)$ induced by $\widetilde{\Omega}^s|_{D^2_\infty}$, which are necessarily loops in $\text{Ham}(M,\omega)$, since $\widetilde{\Omega}^s|_{D^2_0} = \omega$. By the properties of coupling forms in $C_{\max}$, the statement of the lemma is equivalent to γ being a smooth critical point for the L^+ functional on $\Omega\text{Ham}(M,\omega)$. We leave the details to the reader.

Lemma 3.6. *Let $h : B_\gamma \to \Omega Ham(M,\omega)$ be as in Theorem 1.2. Then there is a family $\{\widetilde{\Omega}'_b\} \in C$ on P_h arbitrarily C^∞-close to $\{\widetilde{\Omega}^h_b\}$, with $\widetilde{\Omega}'_{\max} = \widetilde{\Omega}^h_{\max}$, such that the induced family $\{J'_b\}$ is regular for $A_{\max}$-class curves.*

Proof. Denote by $\mathcal{B}$ the space of pairs (u,b), $u \in \mathcal{B}_b$, with $\mathcal{B}_b$ denoting the space of smooth sections of X_b. We have a bundle

$$\mathcal{E} \to \mathcal{B},$$

whose fiber over (u,b) is $\Omega^{0,1}(S^2, (u,b)^* TX_b)$, and the section we call $\mathcal{F}_h$,

$$\mathcal{F}_h(u_b) = \bar{\partial}_{J^h_b}(u).$$

By the assumption that h is Morse at γ, and Proposition 3.4,

$$\mathcal{M}_0(P_h, A_{\max}, \{J^h_b\})$$

is a zero dimensional manifold consisting of a single point $u_{\max}$, which corresponds to the section $\sigma_{x_{\max}}$ of $X_{\max} \subset P_h$. By assumption we have that $I^{virt}(\gamma) = \dim B_\gamma$, where $I^{virt}(\gamma)$ is the cokernel of the vertical differential

$$D\mathcal{F}_h|_{T_{u_{\max}} \mathcal{B}_{\max}}.$$

And so zero is the expected dimension of $\mathcal{M}_0(P_h, A_{\max}, \{J^h_b\})$. Therefore, if we can perturb (abstractly) the section $\mathcal{F}_h$, fixing it over $\mathcal{B}_{\max} \subset \mathcal{B}$, so that the corresponding vertical differential at $u_{\max}$ has no kernel, then the perturbed section would be necessarily transverse to the 0-section at $u_{\max}$. (The corresponding differential would necessarily be Fredholm of index zero.)

More specifically, we need a smooth vertical (tangent to the fibers of $\mathcal{E} \to \mathcal{B}$) vector field $\mathcal{V}$ along $\mathcal{F}_h$ having the properties that it vanishes over $\mathcal{B}_{\max}$, and so that $\mathcal{F}_h$ exponentiated along $\mathcal{V}$,

$$exp_t^{\mathcal{V}}(\mathcal{F}_h) : \mathcal{B} \to \mathcal{E}$$

is transverse to the 0-section for all sufficiently small time t. This is just a matter of differential topology. The assumption that $\mathcal{V}$ vanishes over $\mathcal{B}_{\max}$ can be accommodated due to the fact that the vertical differential $D\mathcal{F}_h$ restricted to $T_{u_{\max}}\mathcal{B}_{\max}$ has no kernel. Which in turn follows from the fact that the vertical normal bundle to $u_{\max}$ in $X_{\max}$ is holomorphic and all its Chern numbers are negative. (Indeed this is another instance where the Morse condition on H is crucial.)

We may also assume that $\mathcal{V}$ vanishes outside a neighborhood $\mathcal{B}_{U_{\max}}$ of the curve $u_{\max}$ in $\mathcal{B}$, with all maps in $\mathcal{B}_{U_{\max}}$ lying over $U_{\max}$ a contractible neighborhood of max in $B_{\max}$. Trivializing P_h over $U_{\max}$ we have $\mathcal{B}_{U_{\max}} = \mathcal{B}_{\max} \times U_{\max}$, where $\mathcal{B}_{\max} = C^\infty(S^2, X_{\max})$.

We now show that the perturbation $\mathcal{V}$ can be realized by perturbing the family $\{\Omega_b^h\}$ and hence the induced family $\{J_b^h\}$. Let

$$\mathcal{E} \to (\mathcal{B}_{\max} \times U_{\max}) \times C_{\max} \equiv \widetilde{C}$$

be the fibration whose fiber over $(u, b, \widetilde{\Omega})$ is $\Omega_J^{0,1}(S^2, u^*TX_{\max})$ the space of j, J-anti-linear one forms, where J is induced by $\widetilde{\Omega}$. Let

$$\mathcal{F} : \widetilde{C} \to \mathcal{E}$$

be the map

$$\mathcal{F}(u, b, \widetilde{\Omega}) = \bar{\partial}_J(u)$$

Denote by $T^{vert}\widetilde{\mathcal{C}}$ the vertical tangent bundle of $pr : \widetilde{C} \to \mathcal{B}_{\max} \times U_{\max}$. The vertical differential

$$D\mathcal{F} : T^{vert}\widetilde{C} \to T^{vert}\mathcal{E},$$

is a family of maps

$$D\mathcal{F}(u, b, \widetilde{\Omega}) : T_{\widetilde{\Omega}}C_{\max} \to \Omega_J^{0,1}(S^2, u^*TX_{\max}). \tag{3.14}$$

By the proof of Theorem 8.3.1 and Remark 3.2.3 in [10], (3.14) is onto for every u, $\widetilde{\Omega}$. (One must of course work with the appropriate Sobolev completions for this.)

Let $\mathcal{S} : U_{\max} \to \widetilde{C}$ be the map $\mathcal{S}(b) = (u_{\max}, b, \widetilde{\Omega}_b^h)$, induced by the family $\{\widetilde{\Omega}_b^h\}$ on P_h over $U_{\max}$. And let

$$\mathcal{A} = D\mathcal{F}^{-1}(\mathcal{V}) \subset T^{vert}\widetilde{C}$$

Since (3.14) is onto for every u, $\widetilde{\Omega}$, $\mathcal{A}|_{\mathcal{S}}$ fibers over $\mathcal{S}$. Let $\mathcal{W}$ be any section, which we may think of as an infinitesimal perturbation of the family $\{\widetilde{\Omega}_b^h\}$ for $b \in U_{\max}$. This perturbation extends by vanishing perturbation outside $U_{\max}$. The infinitesimal perturbation $\mathcal{W}$ is the one we were looking for and so we are done. □

3.2. **Proof of Corollary 1.3.** The first part is immediate. To prove the second part note that $h : B_\gamma \to \Omega\mathrm{Ham}(M,\omega)$ has max length measure $H_{\max}$. On the other hand if the max length measure of the map h could be reduced below $H_{\max}$ by moving it in its homology class to say $h' : B \to \Omega\mathrm{Ham}(M,\omega)$, then this would destroy the contribution to $\Psi(h) = \Psi(h')$ in the c-energy $H_{\max}$, because by Corollary 3.2 there would simply be no vertical $\{J_b^{h'}\}$-holomorphic curves in $P_{h'}$ with c-energy $= H_{\max}$, this is a contradiction. □

3.3. **Proof of Theorem 1.5.** We clearly have $I^H(\gamma) \geq \dim B_\gamma$. Suppose $I^H(\gamma) > \dim B_\gamma = m$, then there a subspace $N \subset T_\gamma \Omega \mathrm{Ham}(M,\omega)$ such that $N \supset h_* T_{\max} B_\gamma$, $\dim N = \dim B_\gamma + 1$ and such that the Hessian of L^+ is negative definite on N. We homotop the map h to a map h' so that $L^+(h'(b)) < L^+(\gamma)$ for all $b \in B_\gamma$, which kills the contribution to $\Psi(h)$ in the energy $H_{\max}$ by the proof of 1.3. This will conclude the proof.

Let $\phi : D^m \to B_\gamma$ be a chart containing max. We may homotop h to a map $\widetilde{h}$ with the same image as h, with $\widetilde{h}$ being the constant map to γ on $\phi(D^m)$. Let $p : D^m \to N - 0$ be an embedding so that

$$p : \partial D^m \to h_*(S^m \subset T_{\max} B_\gamma)$$

is a degree one map, where the unit sphere S^m is determined by the trivialization ϕ. Under the identification given by ϕ extend p to any smooth map

$$\widetilde{p} : B_\gamma \to T\Omega \mathrm{Ham}(M,\omega).$$

Now move $\widetilde{h}$ along $\widetilde{p}$ by exponentiating for a sufficiently small time, then the exponentiated map h' will have the required property. $\square$

4. Morse theory on $\Omega\mathrm{Ham}(G/T)$ and ΩG.

Let $M = G/T$, where T is its maximal torus. There a symplectic structure on G/T, inherited from that of T^*G by symplectic reduction of the natural G action on T^*G, (G/T is the generic leaf of the symplectic reduction.) The leaves of the the symplectic reduction of T^*G and hence G/T can be identified with orbits of the coadjoint action of G on $\mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G. The symplectic structure is then induced from a natural 2 form on $\mathfrak{g}^*$ called the Kirillov form, (see [1]).

Let G be semisimple and $\mathcal{O}_{p_0}$ a coadjoint orbit of $p_0 \in \mathfrak{g}^*$ by G. Then G acts on $\mathcal{O}_{p_0}$ by $\phi_g(p) = Ad^*_{g^{-1}}(p)$. With the infinitesimal action $X_\eta(p) = -ad^*_\eta(p)$ for $\eta \in \mathfrak{g}$. The generating function is defined by $H_\eta(p) = p(\xi)$ and it is normalized, as the map

$$\eta \mapsto \int_{O_\xi} H_\eta(p) = \int_{O_\xi} p(\eta)$$

defines an element of $\mathfrak{g}^*$, which is clearly invariant under the coadjoint action of G and so must be 0 (since $\mathfrak{g}$ has no center).[1]

Suppose now $\xi = \frac{d}{dt}|_0 \gamma \in \mathfrak{g}$, where γ is a one parameter subgroup and let $\mathcal{O}_\xi$ denote the coadjoint orbit of the covector $< \frac{\xi}{||\xi||}, \cdot >$. In this case the maximum of the generating function H_ξ on $\mathcal{O}_\xi$ is $< \frac{\xi}{||\xi||}, \xi >= ||\xi||$. Moreover, we get an inequality relating positive Hofer norm with Riemmanian norm,

$$||\eta||^+ \leq ||\eta||, \tag{4.1}$$

for any η, where

$$||\eta||^+ = \max_{\mathcal{O}_\xi} H_\eta.$$

In this discussion the symplectic manifold $\mathcal{O}_\xi$ depends on ξ. If we make an additional assumption, that the subgroup of G fixing $< \xi, \cdot >$, under the coadjoint action is T, then we can identify $O_\xi \simeq (G/T, \omega_\xi)$. Moreover, this condition is generic in $\mathfrak{g}^*$, from which it follows that the symplectic forms ω_ξ are deformation equivalent. Also

[1] I would like to thank Yael Karshon for suggesting this argument.

$(G/T, \omega_\xi, j_\xi)$, are Fano for an integrable complex structure j_ξ depending smoothly on ω_ξ. Therefore, we may regard $\mathcal{O}_\xi$ as simply G/T for our purposes of quantum homology and the map Ψ, (cf. Remark 2.2)

4.1. **Morse theory on ΩG.** Let $h : B_\gamma \to \Omega G$ be the pseudocycle corresponding to the unstable manifold of γ for the energy flow on ΩG. (See the discussion in Section 1.4. It is necessarily a pseudocycle, since all the indexes of critical points of E are even.) As before we denote by $\max \in B_\gamma$ the point $h(\max) = \gamma$.

Theorem 4.1. *Let G be a semi simple compact Lie group, then the positive Hofer length functional $L^+ : B_\gamma \to \mathbb{R}$ (its pullback from $\Omega Ham(O_\xi)$) is Morse at* max. *Moreover, if the centralizer of γ is the torus then the indexes $I^{virt}(f(\gamma))$ (cf. eq.* (1.3)*) and the Riemannian index of γ coincide. In other words:*

$$I^{virt}(f(\gamma)) = \dim B_\gamma. \tag{4.2}$$

Proof. By (4.1) we have $L^+(\gamma_b) \le L(\gamma_b)$ for γ_b any loop in B_γ (or ΩG), where L is the Riemmanian length functional on ΩG. Since $L^+(\gamma) = L(\gamma)$, the first part of the theorem will follow if the restriction of L to B_γ is Morse at γ. This is intuitively clear as the restriction of the energy functional E to B_γ is Morse at γ since E is a Morse-Bott function on ΩG. Here are the details. Let γ_t be a smooth variation of $\gamma = \gamma_0$ in B_γ. Applying Cauchy-Schwarz inequality,

$$\left(\int_a^b fg\, d\theta\right)^2 \le \left(\int_a^b f^2 d\theta\right)\left(\int_a^b g^2\, d\theta\right),$$

with $f(\theta) = 1$ and $g(\theta) = ||\frac{d}{d\theta}|_\theta \gamma_t(\theta)||$, we get

$$L(\gamma_t)^2 \le E(\gamma_t),$$

since γ is parametrized from 0 to 1. Both sides are the same for $t = 0$, (since γ is a geodesic and so parametrized by arclength) and the derivatives of both sides are 0 at $t = 0$ since γ is critical for both L and E. It follows that

$$\frac{d^2}{dt^2}|_0 L(\gamma_t)^2 = 2L(\gamma) \cdot \frac{d^2}{dt^2}|_0 L(\gamma_t) \le \frac{d^2}{dt^2}|_0 E < 0,$$

and so

$$\frac{d^2}{dt^2}|_0 L(\gamma_t) < 0.$$

We now prove the second part of the theorem. Let γ be generic and ξ the corresponding element in $\mathfrak{g}$. In order to compute $I^{virt}(f \circ \gamma)$ we need to understand the weights of the coadjoint action of γ on the tangent space $T_p O_\xi$, where p is the maximal fixed point $p = < \frac{\xi}{||\xi||}, \cdot >$. Since the maps Ad^*_g are linear, this action can be identified with the action of γ on a subspace of $T_0\mathfrak{g}^* \equiv \mathfrak{g}^*$. Moreover, under the identification of $\mathfrak{g}^*$ with $\mathfrak{g}$ using the Ad-invariant inner product $<,>$ on $\mathfrak{g}$ the coadjoint action by Ad^*_g on $\mathfrak{g}^*$ corresponds to the adjoint action by $Ad_{g^{-1}}$ on $\mathfrak{g}$ and so the coadjoint action of γ on $\mathfrak{g}^*$ corresponds to the adjoint action of γ^{-1} on $\mathfrak{g}$. More specifically, we want the adjoint action on a certain subspace of $T_p \subset \mathfrak{g}$ which corresponds under all these identifications to $T_p O_\xi$. In fact this subspace can be determined synthetically as follows. Write

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha, \tag{4.3}$$

where $\mathfrak{t}$ is the maximal Abelian subalgebra of $\mathfrak{g}$ containing ξ and $\mathfrak{g}_\alpha$ is a subspace of $\mathfrak{g}$ on which γ^{-1} is acting by $e^{\alpha 2\pi i\theta}$. (So that $\mathfrak{t}$ corresponds to $\alpha = 0$.) Now, T_p is invariant under the adjoint action of γ^{-1} and all the weights α are necessarily non zero on T_p and are negative. The latter is due to the fact that the function H_ξ on O_ξ is Morse at its maximum p, which together with our convention $X_H = -J \operatorname{grad} H$ implies that the weights are negative. The subspace T_p must then simply be

$$T_p = \bigoplus_\alpha \mathfrak{g}_\alpha.$$

The virtual index is then by definition

$$\sum_\alpha 2|\alpha| - 2.$$

Using the index theorem in Riemannian geometry one can show that this is the Riemannian index of the geodesic γ of G, see for example proof of Bott periodicity, [11, Section 23]. $\square$

Corollary 4.2. *Let G be a semi simple Lie group and γ a generic S^1 subgroup. Then the pseudocycles $f \circ h : B_\gamma \to \Omega Ham(G/T)$ satisfy the hypothesis of Theorem 1.2.*

Remark 4.3. *Strictly speaking the map Ψ in Theorem 1.2 is only defined here and in* [16] *on cycles $f : B \to \Omega Ham(M, \omega)$, where B is a closed smooth manifold. However there is no essential difficulty in extending this to appropriate pseudocycles. The details of this will be given in a more general context in the upcoming paper* [15].

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